Time-Dependent Stresses in Prestressed Concrete Sections of General Shape

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A procedure is presented for determining stresses and strains in reinforced concrete sections with or without prestressing. Cross sections of arbitrary shape are considered and the immediate and long term stresses and strains are evaluated, accounting for the effects of creep and shrinkage of concrete and relaxation of prestressed steel. A numerical example is included to show how the proposed method can be applied.

An accurate determination of stresses and strains due to time-dependent effects is important in the analysis and design of prestressed concrete structures. These volume changes, caused by creep and shrinkage of concrete and relaxation of prestressed steel, manifest themselves in the calculation of prestress losses. This paper presents a rigorous method for determining time-dependent stresses in prestressed concrete sections of general shape.

Fig. 1 shows a general shape of a cross section of a reinforced concrete member subjected to a normal force N at a reference point O and moments M_x and M_y about orthogonal axes x and y. The position of the reference point and the directions of the orthogonal axes are arbitrary. The forces N, M_x and M_y are assumed to be introduced at time t_o. The values of N, M_x and M_y include the prestressing contributions: $-P$, $-Py_p$ and $-Px_p$, respectively, where P is the absolute value of a prestress force introduced at time t_o, and x_p and y_p are the coordinates of the prestressing tendon.

The problem treated here is to determine the stress and strain distributions at time t_o and at a later instant t after occurrence of creep, shrinkage of concrete and relaxation of prestressed steel. Examples of practical cross section shapes for which the procedure of analysis presented can be applied are shown in Fig. 2; these shapes are frequently used for concrete piles, poles, bridge piers, box girders, chimneys, shells and shear walls or corner columns in buildings.

The forces are shown in Fig. 1 in their positive directions. The loss of prestress due to creep, shrinkage and relaxation does not need to be estimated separately, as is done in conventional analysis of prestressed concrete structures. The analytical procedure presented below gives the time-dependent change of the value of P in the period t_o to t. Application of the same procedure for sections having one plane of symmetry, subjected to N and M_x (with $M_y = 0$), is treated by Ghali and Favre.[1]

Because member cross sections subjected to a normal force and biaxial bending are often encountered in practice, the problem of calculating the stress and strain before and after cracking and determining the ultimate strength have been extensively treated.[2-12] The present paper, and a companion paper,[13] offer a more complete solution to the problem. The time-dependent changes in stress and strain due to creep, shrinkage and relaxation are analyzed.

In the companion paper, after occurrence of the time-dependent changes, the section is subjected to incremental forces ΔN, ΔM_x and ΔM_y, representing the effects of live load, wind, earthquake, and other loads of increasing magnitude up to failure. The analysis allows the tracing of the variation of deformations (axial strain and curvatures) beyond the ultimate load; this information is needed in modern design for earthquakes, where both the strength and ductility are of concern. Any stress-strain (σ-ε) relation can be used for concrete, prestressed steel and nonprestressed steel. Thus, the ductility can be assessed when new materials, such as high strength concrete, are used.

The analysis in this paper and the companion paper[13] is given in sufficient detail to allow preparation of a single computer program for use in practice for reinforced concrete cross sections of general shape, with or without prestressing. The computer program can be used to satisfy the design requirements for serviceability, ultimate strength, and ductility.

Computer programs that perform the analyses in this paper and the companion paper[13] can be made available by contacting the authors.

In this paper, it is assumed that the concrete stresses are not too high, thereby allowing use of a linear stress-strain relation and ignoring cracking. The case where the tensile stresses produce cracking of concrete and calculation of ultimate strength of prestressed cross sections are treated in the companion paper.[13]

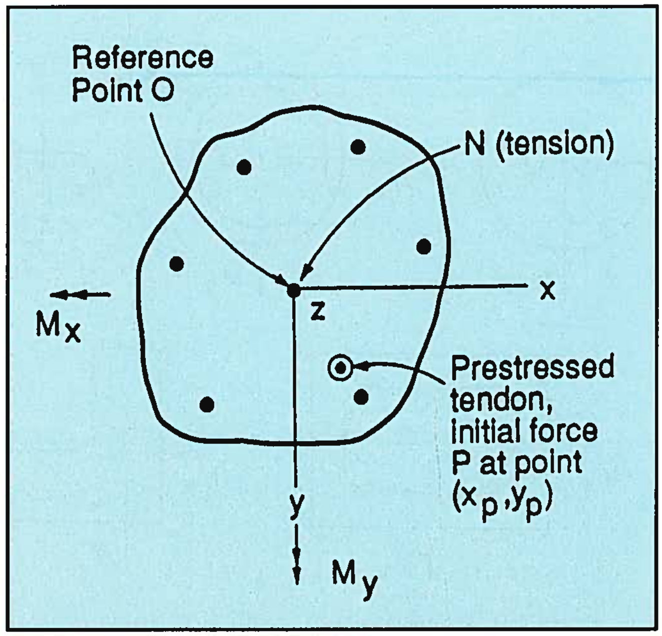

Fig. 1. General cross section shape of reinforced concrete member. Positive sign convention.

Adrian and Triantafillou (1992)[16] treated the same problem for cross sections that can be described in rectangular elements.

Recently, a paper by Kawakami et al. (1996)[17] has been published in Japanese. The article analyzes different cross sections but essentially uses the same analytical approach as the present paper.

STRESS-STRAIN RELATIONS FOR CONCRETE

A stress increment $\Delta\sigma_c$ introduced at time t_o and sustained, without change in magnitude, produces at time t a strain given by:

$$\varepsilon_c(t) = \frac{\Delta\sigma_c}{E_c(t_o)}(1+\varphi) \qquad (1)$$

where $E_c(t_o)$ is the modulus of elasticity of concrete at the instant t_o and $\varphi[= \varphi(t,t_o)]$ is the creep coefficient; the value of φ representing the ratio of creep to the instantaneous strain depends on the properties of the concrete and the environment in which it is kept. Values of φ are proposed by ACI 209[14] and the FIP-CEB Model Code[15] (see Ref. 1).

When the stress increment $\Delta\sigma_c$ is gradually introduced from zero at time t_o to its full value at time t, the strain at time t is given by:

$$\varepsilon_c(t) = \frac{\Delta\sigma_c}{E_c(t_o)}(1+\chi\varphi) \qquad (2)$$

where χ [= $\chi(t,t_o) \cong 0.8$] is the aging coefficient of concrete. The values of χ determined by numerical procedure are presented in the form of graphs (Ref. 1).

Eq. (2) is rewritten in the form:

$$\varepsilon_c(t) = \frac{\Delta\sigma_c}{\bar{E}_c} \qquad (3)$$

$\bar{E}_c$ [= $\bar{E}_c(t,t_o)$] is the age-adjusted elasticity modulus defined by:

$$\bar{E}_c = \frac{E_c(t_o)}{1+\chi\varphi} \qquad (4)$$

For analysis of the stress and strain occurring immediately after load application on reinforced concrete sections, the term "transformed cross section" is used to mean a section composed of the area of concrete A_c plus the areas of the nonprestressed and prestressed steels A_{ns} and A_{ps}, respectively, multiplied by:

$$\alpha_{ns} \text{ or } \alpha_{ps} = \frac{E_{ns} \text{ or } E_{ps}}{E_c(t_o)} \qquad (5)$$

where E_{ns} or E_{ps} is the modulus of elasticity of the nonprestressed steel or

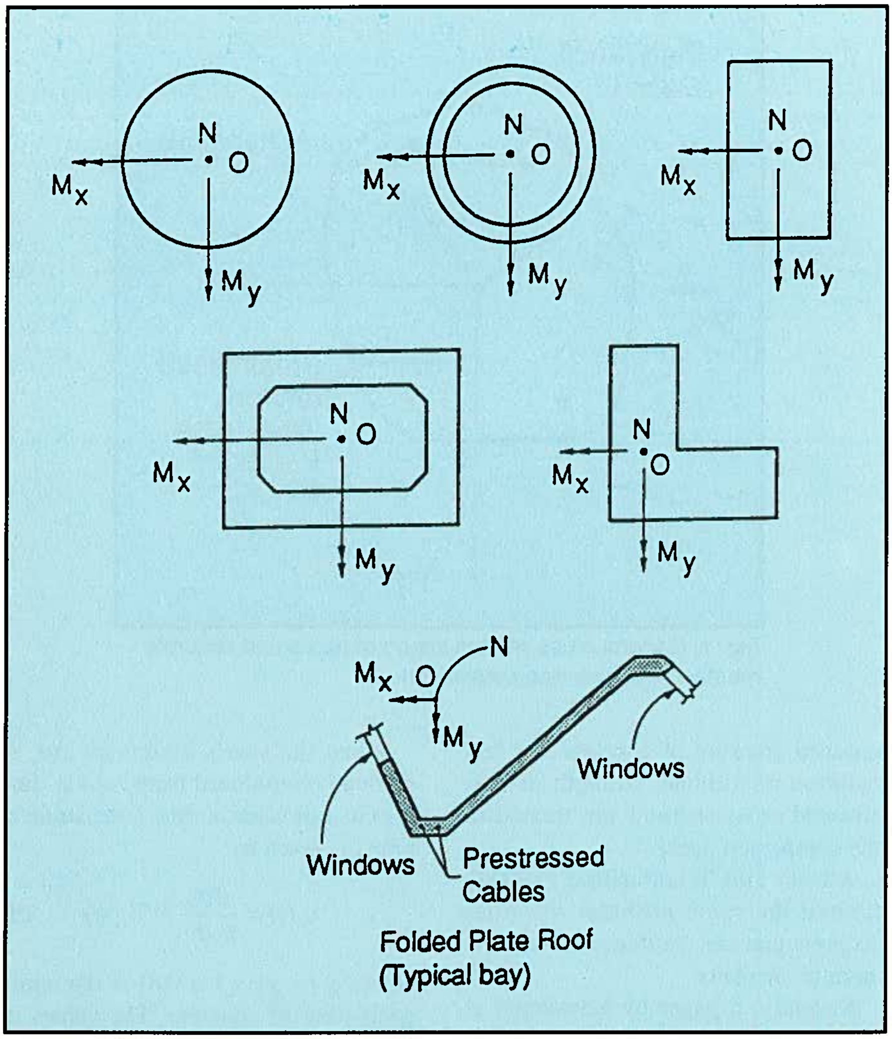

Fig. 2. Practical shapes of cross sections. Location of reference Point O is arbitrary.

the prestressed steel, respectively.

When the analysis is for changes in stress and strain due to creep, shrinkage and relaxation, the term "age-adjusted transformed section" is used. This is composed of the area of concrete A_c plus the areas of the nonprestressed and prestressed steel A_{ns} and A_{ps}, respectively, multiplied by:

$$\bar{\alpha}_{ns} \text{ or } \bar{\alpha}_{ps} = \frac{E_{ns} \text{ or } E_{ps}}{\bar{E}_c(t_o)} \qquad (6)$$

BASIC MECHANICS EQUATIONS

In this section, basic mechanics equations for the stresses and strains in homogenous cross sections are reviewed; they will be applied in the following sections in various steps of the analysis procedure.

The widely accepted hypothesis that plane cross sections remain plane allows that the strain at any point be presented by (see Fig. 1):

$$\varepsilon = \varepsilon_O + \psi_x y + \psi_y x \qquad (7)$$

where ε_O is the strain at the reference point O; $\psi_x = \partial\varepsilon/\partial y$ and $\psi_y = \partial\varepsilon/\partial x$ represent the curvatures in the yz and xz planes.

With a linear stress-stain relation, the stress at any point and the stress resultants are expressed as:

$$\sigma = E\varepsilon \qquad (8)$$

$$N = \int \sigma dA \qquad (9)$$

$$M_x = \int \sigma y\, dA \qquad (10)$$

$$M_y = \int \sigma x\, dA \qquad (11)$$

Substitution of Eq. (7) in Eqs. (8) to (11) gives:

$$N = E(A\varepsilon_O + B_x\psi_x + B_y\psi_y) \qquad (12)$$

$$M_x = E(B_x\varepsilon_O + I_x\psi_x + I_{xy}\psi_y) \qquad (13)$$

$$M_y = E(B_y\varepsilon_O + I_{xy}\psi_x + I_y\psi_y) \qquad (14)$$

where B and I represent first and second moments of area about the x or y axes:

$$B_x = \int y\, dA;\ B_y = \int x\, dA \qquad (15)$$

$$I_x = \int y^2 dA;\ I_y = \int x^2 dA \qquad (16)$$

$$I_{xy} = \int xy\, dA \qquad (17)$$

Eqs. (12) to (14) can be written in matrix form:

$$\begin{Bmatrix} N \\ M_x \\ M_y \end{Bmatrix} = E \begin{bmatrix} A & B_x & B_y \\ B_x & I_x & I_{xy} \\ B_y & I_{xy} & I_y \end{bmatrix} \begin{Bmatrix} \varepsilon_O \\ \psi_x \\ \psi_y \end{Bmatrix} \qquad (18)$$

This equation can be used to determine the stress resultants N, M_x and M_y when the strain or stress distributions are known. The inverse of Eq. (18) can be used to determine for given values of N, M_x and M_y, the axial strain and the curvatures that define the strain distribution [Eq. (7)]:

$$\begin{Bmatrix} \varepsilon_O \\ \psi_x \\ \psi_y \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} A & B_x & B_y \\ B_x & I_x & I_{xy} \\ B_y & I_{xy} & I_y \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_x \\ M_y \end{Bmatrix} \qquad (19)$$

Multiplication of the matrix by E gives three stress parameters defining the stress distribution:

$$\sigma_O = E\varepsilon_O;\ \gamma_x = E\psi_x;\ \gamma_y = E\psi_y \qquad (20)$$

The stress at any point is:

$$\sigma = \sigma_O + \gamma_x y + \gamma_x x \qquad (21)$$

where $\gamma_x = \partial\sigma/\partial y$ and $\gamma_y = \partial\sigma/\partial x$ (slopes of the stress diagram).

The intercepts of the neutral axis with the x and y axes are, respectively:

$$a = -\frac{\sigma_O}{\gamma_y};\ b = -\frac{\sigma_O}{\gamma_x} \qquad (22)$$

The resultants of the stress defined by Eq. (21) are:

$$\begin{Bmatrix} N \\ M_x \\ M_y \end{Bmatrix} = \begin{bmatrix} A & B_x & B_y \\ B_x & I_x & I_{xy} \\ B_y & I_{xy} & I_y \end{bmatrix} \begin{Bmatrix} \sigma_O \\ \gamma_x \\ \gamma_y \end{Bmatrix} \qquad (23)$$

The equations presented in this section apply to reinforced concrete cross sections by using the area properties A, B and I of the transformed or the age-adjusted transformed section, as will be indicated below.

Evaluation of the area properties of

sections of general shape can be performed by the numerical procedure given in Appendix A. The procedure, suitable for automatic computation, is particularly useful when cracking is considered and concrete in tension is ignored. In this case, the calculation requires iteration to determine the neutral axis and the area properties will have to be determined in each iteration.[13]

FOUR ANALYSIS STEPS

Four steps can be followed to determine the stress and strain distributions at time t_o, immediately after application of N, M_x and M_y (see Fig. 1) and at time t after occurrence of creep, shrinkage and relaxation.

Step 1

Apply N, M_x and M_y on a transformed section composed of A_c plus $(\alpha_{ps}A_{ps} + \alpha_{ns}A_{ns})$. The transformed section includes only the prestressed and nonprestressed steel bonded to the concrete at prestress transfer. Thus, A_{ps} should be included when pretensioning is used, but when all the prestressing steel is post-tensioned in one stage, A_{ps} and the area of the duct should be excluded.

When the structure is statically indeterminate, the indeterminate normal force and moments should be included in the values of N, M_x and M_y.

Apply Eq. (19) to determine $\varepsilon_O(t_o)$, $\psi_x(t_o)$ and $\psi_y(t_o)$, which define the distribution of the instantaneous strain [Eq. (7)]. Multiplication by $E_c(t_o)$ gives $\sigma_O(t_o)$, $\gamma_x(t_o)$ and $\gamma_y(t_o)$, which define the distribution of the instantaneous concrete stress [Eqs. (20) and 21)]. Multiplication of the concrete stress by $\alpha_{ns}(t_o)$ gives the stress in the nonprestressed steel.

When pretensioning is used, the stress in the prestressed steel immediately after prestress transfer is equal to the concrete strain, $\varepsilon(t_o)E_{ps}$, plus the initial tension. When post-tensioning is used, the stress in the prestressed steel at t_o is simply equal to the initial tension.

Step 2

Determine the hypothetical change, in the period t_o to t, in the strain distribution due to creep and shrinkage if they were free to occur. The strain change at Point O is equal to $[\varphi(t,t_o)\,\varepsilon_O(t_o) + \varepsilon_{cs}]$ and the changes in curvatures are: $[\varphi(t,t_o)\,\psi_x(t_o)]$ and $[\varphi(t,t_o)\,\psi_y(t_o)]$; where ε_{cs} is the free shrinkage of concrete in the period t_o to t. It is assumed here that ε_{cs} has a constant value over the section.

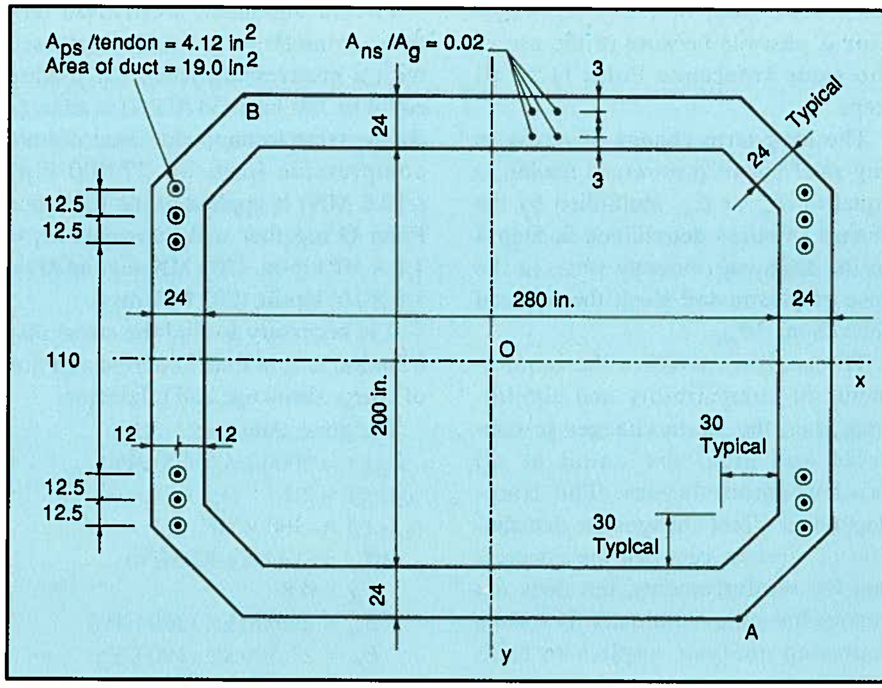

Fig. 3. Prestressed concrete cross section of a bridge pier.

Step 3

Calculate the artificial stress that, when gradually introduced in the concrete during the period t_o to t, will prevent occurrence of the strain determined in Step 2. The restraining stress at any point is [see Eqs. (3) and (7)]:

$$\Delta\sigma_{restrained} = -\bar{E}_c[\varepsilon_{cs} + \varphi(t,t_o)\{\varepsilon_O(t_o) + \psi_x(t_o)y + \psi_y(t_o)x\}] \tag{24}$$

This stress distribution is defined by a stress value at Point O and two γ values [Eq. (21)]. The three stress parameters are equal to $(-\bar{E}_c)$ multiplied by the three strain parameters determined in Step 2.

Step 4

Determine by Eq. (18) a force at Point O and two moments, which are the resultants of $\Delta\sigma_{restrained}$.

The change in concrete strain due to relaxation of the prestressed steel can be artificially prevented by the application, at the level of the prestressed steel, of a restraining force equal to $A_{ps}\Delta\bar{\sigma}_{pr}$, where $\Delta\bar{\sigma}_{pr}$ is the reduced value* of stress relaxation in the period t_o to t. Substitute the restraining force by a force of the same magnitude at Point O plus moments about the x and y axes. Summing up gives $\{\Delta N, \Delta M_x, \Delta M_y\}_{restrained}$, the restraining forces required to artificially prevent the strain change due to creep, shrinkage and relaxation.

To eliminate the artificial restraint, apply $\{\Delta N, \Delta M_x, \Delta M_y\}_{restrained}$ in reversed directions on an age-adjusted transformed section composed of A_c plus $\bar{\alpha}_{ps}A_{ps} + \bar{\alpha}_{ns}A_{ns}$, calculate the corresponding changes in strains and stresses by Eqs. (19) and (20).

The concrete strain distribution at time t is the sum of the strains determined in Steps 1 and 4; the corresponding stress is the sum of the stress at t_o calculated in Step 1 and the time-dependent changes calculated in Steps 3 and 4. Superposition of strains or stresses in the various steps can be done by summing up the increments

* The reduced relaxation of prestressed steel is equal to the intrinsic relaxation multiplied by a reduction factor equal to 0.8 approximately; a more accurate value can be determined (see Ref. 1). The intrinsic relaxation is the time-dependent change in stress (a negative value) in a tendon stretched between two fixed points.

$\{\Delta\varepsilon_O, \Delta\psi_x, \Delta\psi_y\}$ or $\{\Delta\sigma_O, \Delta\gamma_x, \Delta\gamma_y\}$. This is possible because of the use of the same reference Point O in all steps.

The long-term change of stress in any steel bar or prestressed tendon is equal to $\bar{\alpha}_{ns}$ or $\bar{\alpha}_{ps}$ multiplied by the change of stress determined in Step 4 in the adjacent concrete plus, in the case of prestressed steel, the reduced relaxation, $\Delta\bar{\sigma}_{pr}$.

The analysis satisfies the requirements of compatibility and equilibrium, i.e., the strain changes in concrete and steel are equal at all reinforcement layers. The time-dependent effect changes the distribution of stresses between the concrete and the reinforcements, but does not change the stress resultants. The same four-step analysis applies to reinforced concrete sections without prestressing, simply by setting $A_{ps} = 0$. The same procedure can also be used for the analysis of composite sections, made of more than one type of concrete cast, prestressed in stages, or made of concrete and structural steel (see Ref. 1).

IMMEDIATE AND LONG-TERM DEFLECTIONS

The curvatures ψ_x and ψ_y at time t_o and at time t can be used to give the immediate or long-term deflections at any section of a concrete structure by virtual work:

$$D = \int \psi_x M_{ux} dl + \int \psi_y M_{uy} dl \quad (25)$$

where M_{ux} and M_{uy} are the bending moments due to a unit load applied at the location and direction in which the deflection is to be calculated. The integrals can be evaluated numerically, over the length of each member of the structure using curvature values at a limited number of sections (e.g., three per member)(see Ref. 1).

NUMERICAL EXAMPLE

Fig. 3 shows the cross section of a prestressed concrete bridge pier. The nonprestressed steel is uniformly arranged over the perimeter and its cross-sectional area A_{ns} amounts to $0.02A_g$, where A_g is the gross concrete area.

Twelve unbonded prestressed tendons, symmetrically located, are used with a prestressing force per tendon equal to 780 kips (3470 kN) at time t_o. At the same instant, additional normal compressive force of -27,000 kips (-12.0 MN) is applied at the reference Point O together with moments $M_x = 1.8 \times 10^6$ kip-in. (203 MN-m) and $M_y = 1.8 \times 10^6$ kip-in. (203 MN-m).

It is necessary to find the stress distribution at t_o at time t after occurrence of creep, shrinkage and relaxation.

The given data are:

$E_c(t_o) = 5000$ ksi (34.5 GPa)
$\varphi(t,t_o) = 2.1$
$\varepsilon_{cs}(t,t_o) = -300 \times 10^{-6}$
$\Delta\bar{\sigma}_{pr} = -12$ ksi (-83 MPa)
$\chi = 0.8$
$E_{ns} = 29{,}000$ ksi (200 GPa)
$E_{ps} = 27{,}500$ ksi (190 GPa)

Properties of the transformed section at time t_o:

$$\alpha_{ns} = \frac{E_{ns}}{E_c(t_o)} = \frac{29{,}000}{5000} = 5.8$$

The transformed section can be considered to be composed of the gross area A_g minus the prestress ducts plus $0.02A_g(\alpha_{ns} - 1)$. This gives the following properties:

$A = 25{,}270$ in.^{2} (16.30 m^2)
$I_x = 216.2 \times 10^6$ in.^{4} (89.97 m^4)
$I_y = 331.2 \times 10^6$ in.^{4} (137.8 m^4)
$B_x = B_y = 0$

The age-adjusted modulus of elasticity of concrete [Eq. (4)] is obtained from:

$$\bar{E}_c(t,t_o) = \frac{5000}{1+0.8(2.1)} = 1866 \text{ ksi}$$

$$\bar{\alpha}_{ns} = \frac{29{,}000}{1866} = 15.54$$

$$\bar{\alpha}_{ps} = \frac{275{,}000}{1866} = 14.74$$

The age-adjusted transformed section can be considered to be composed of the concrete area, A_g, plus $0.02A_g$ $(\bar{\alpha}_{ns} - 1)$ plus $(\bar{\alpha}_{ps} - 1)A_{ps}$. This gives the following properties:

$\bar{A} = 30{,}700$ in.^{2} (19.81 m^2)
$\bar{I}_x = 259.3 \times 10^6$ in.^{4} (107.9 m^4)
$\bar{I}_y = 412.2 \times 10^6$ in.^{4} (171.6 m^4)
$\bar{B}_x = \bar{B}_y = 0$

The four steps described above are followed below.

Step 1

$N = -27{,}000 - 12(780)$
$= -36{,}360.0$ kips (161700 kN)
$M_x = 1.8 \times 10^6$ kip-in. (203 MN-m)
$M_y = 1.8 \times 10^6$ kip-in. (203 MN-m)

The parameters defining the strain distribution at time t_o are [Eq. (19)]:

$$\varepsilon_O(t_o) = \frac{N}{E_c(t_o)A} = \frac{-36{,}360}{5000(25{,}270)} = -288 \times 10^{-6}$$

$$\psi_x(t_o) = \frac{M_x}{E_c(t_o)I_x} = \frac{1.8\times10^6}{5000\left(216.2\times10^6\right)} = 1.665\times10^{-6} \text{ in.}^{-1}$$

$$\psi_y(t_o) = \frac{M_y}{E_c(t_o)I_y} = \frac{1.8\times10^6}{5000\left(331.2\times10^6\right)} = 1.087\times10^{-6} \text{ in.}^{-1}$$

Substitution in Eqs. (20) and (21) gives the stress distribution in concrete at time t_o [see Fig. 4(a)]:

$$\sigma(t_o) = (-1.439 + 8.325 \times 10^{-3}y + 5.435 \times 10^{-3}x) \text{ ksi}$$

The intercepts of the neutral axis with the x and y axes are [Eq. (22)]: $a(t_o) = 264.8$ in. (6.726 m); $b(t_o) = 172.9$ in. (4.392 m). The maximum and minimum stresses are 0.245 and -3.123 ksi (1.69 and -21.4 MPa), respectively, at Points A and B (see Fig. 3).

At time t_o, immediately after prestressing, the stress in the nonprestressed steel is equal to $\alpha_{ns}(t_o)\sigma(t_o) = 5.8\,\sigma(t_o)$; this gives:

$$\sigma_{ns}(t_o) = -8.346 + 48.29 \times 10^{-3}y + 31.52 \times 10^{-3}x \text{ ksi}$$

The stress in the prestressed steel at the same instant is equal to the initial tension:

$$\sigma_{ps}(t_o) = \frac{780}{4.12} = 189.3 \text{ ksi (1305 MPa)}$$

Step 2

The distribution of the strain that would occur if shrinkage and creep were unrestrained is defined by the strain change at Point O = $2.1(-288 \times 10^{-6}) - 300 \times 10^{-6} = -905 \times 10^{-6}$ and the changes in curvatures = $2.1\,(1.665 \times 10^{-6}) = 3.497 \times 10^{-6}$ in.$^{-1}$ and $2.1(1.087 \times 10^{-6}) = 2.283 \times 10^{-6}$ in.$^{-1}$

Step 3

The distribution of artificial stress to restrain shrinkage and creep is given by [Eq. (24)]:

$\Delta\sigma_{restrained}$
$= -1866(-905 + 3.497y + 2.283x)10^{-6}$
$= (1.689 - 6.525 \times 10^{-3}y - 4.260 \times 10^{-3}x)$ ksi

Step 4

The concrete cross section properties, including grouted prestressed ducts, are:

$A_c = 22{,}750$ in.^{2} (14.68 m^2)
$I_{cx} = 193.9 \times 10^6$ in.^{4} (80.71 m^4)
$I_{cy} = 299.6 \times 10^6$ in.^{4} (124.7 m^4)

The resultants of $\Delta\sigma_{restrained}$ are found from Eq. (18):

$(\Delta N)_{creep + shrinkage} = (22{,}750) \times (1.689) = 38{,}420$ kips (170.9 MN)

$(\Delta M_x)_{creep + shrinkage} = (193.9 \times 10^6) \times (-6.525 \times 10^{-3}) = -1265 \times 10^3$ kip-in. (-142.9 MN-m)

$(\Delta M_y)_{creep + shrinkage} = (299.6 \times 10^6) \times (-4.260 \times 10^{-3}) = -1276 \times 10^3$ kip-in. (-144.2 MN-m)

$(\Delta N)_{relaxation} = 12(4.12)(-12) = -593$ kips (-2638 kN)

No moments are required to restrain relaxation. Sum up and apply the restraining forces in reversed direction on the age-adjusted transformed section to obtain the stress parameter increments [Eqs. (19) and (20)]:

$$\Delta\sigma_O = \frac{1}{30{,}700}(-38{,}420 + 593) = -1.232 \text{ ksi}$$

$$\Delta\gamma_x = \frac{1}{259.3 \times 10^6}\left(1265 \times 10^3\right) = 4.879 \times 10^{-3} \text{ ksi/in.}$$

$$\Delta\gamma_y = \frac{1}{412.1 \times 10^6}\left(1276 \times 10^3\right) = 3.096 \times 10^{-3} \text{ ksi/in.}$$

The stress in concrete at time t is the sum of the stress determined in Steps 1, 3 and the present step [see Fig. 4(b)]:

$$\begin{aligned}\sigma(t) &= -1.439 + 1.689 - 1.232 \\ &\quad + (8.325 - 6.525 + 4.879)10^{-3}y \\ &\quad + (5.435 - 4.260 + 3.096)10^{-3}x \\ &= -0.982 + 6.679 \times 10^{-3}y + 4.271 \times 10^{-3}x \text{ ksi}\end{aligned}$$

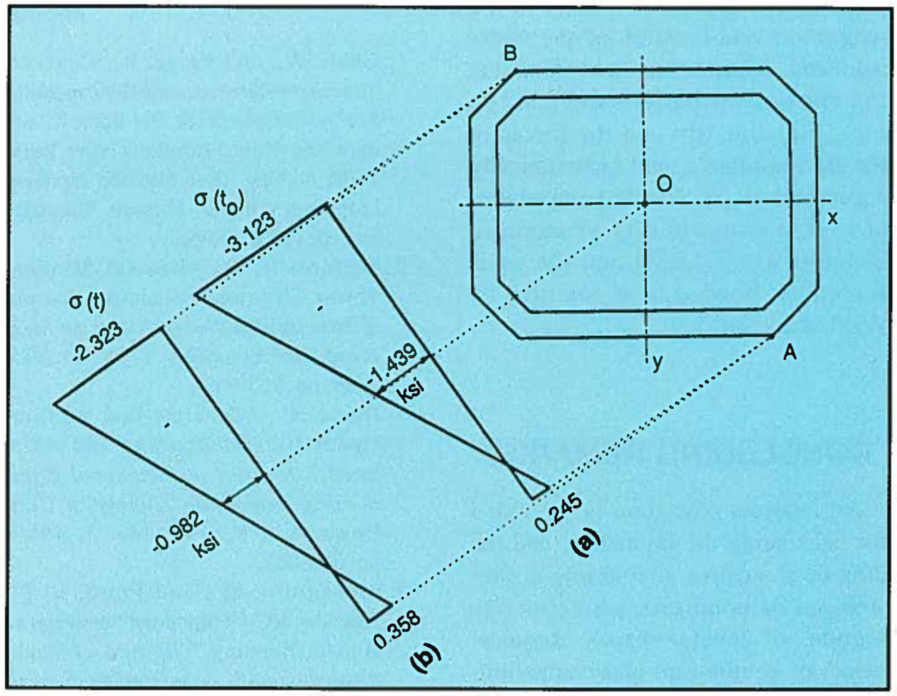

Fig. 4. Stresses in concrete at time t_o and t in example.

Intercepts of the neutral axis at time t with the x and y axes are from Eq. (22):

$a(t) = 229.9$ in. (5.839 m)
$b(t) = 147.0$ in. (3.733 m)

The axial strain and the curvatures at time t are the sum of the values determined in Step 1 and the present step [= the stress parameters ($\Delta\sigma_o$, $\Delta\gamma_x$, $\Delta\gamma_y$) divided by $\bar{E}_c$]:

$$\varepsilon_O(t) = -287.8 \times 10^{-6} - 1.232/1866 = -948.0 \times 10^{-6}$$

$$\psi_x(t) = 1.665 \times 10^{-6} + 4.879 \times 10^{-3}/1866 = 4.280 \times 10^{-6} \text{ in.}^{-1}$$

$$\psi_y(t) = 1.087 \times 10^{-6} + 3.096 \times 10^{-3}/1866 = 2.746 \times 10^{-6} \text{ in.}^{-1}$$

The stress in the nonprestressed steel at time t is the sum of $\sigma_{ns}(t_o)$ and $\bar{\alpha}_{ns}$ times the change in concrete stress calculated in the present step:

$$\sigma_{ns}(t) = -8.346 + 48.29 \times 10^{-3}y + 31.52 \times 10^{-3}x + 15.54(-1.232 + 4.879 \times 10^{-3}y + 3.096 \times 10^{-3}x) \text{ ksi}$$

or

$$\sigma_{ns}(t) = -27.49 + 124.1 \times 10^{-3}y + 79.63 \times 10^{-3}x \text{ ksi}$$

The stress in the prestressed steel at time t is the sum of the initial tension, the reduced relaxation $\Delta\bar{\sigma}_{pr}$ and $\bar{\alpha}_{ps}$ times the change in concrete stress calculated in the present step:

$$\sigma_{ps}(t) = +189.3 - 12.0 + 14.74(-1.232 + 4.879 \times 10^{-3}y + 3.096 \times 10^{-3}x) \text{ ksi}$$

or

$$\sigma_{ps}(t) = 159.1 + 71.92 \times 10^{-3}y + 45.64 \times 10^{-3}x \text{ ksi}$$

The stresses in concrete at time t at Points A and B [see Fig. 4(b)] are 0.358 and -2.323 ksi (2.469 and -16.02 MPa), respectively. The compressive stress at the extreme fiber dropped from -3.123 at time t_o to -2.323 at time t (25 percent) due to creep, shrinkage and relaxation.

Long-term stresses for prestressed concrete structures are often determined by considering a plain concrete section subjected to the sustained forces and a reduced prestressing force; the reduction represents the loss of force in the prestressed steel due to creep, shrinkage and relaxation. Such an analysis ignores the presence of the nonprestressed steel and does not ensure compatibility of strains between concrete and all layers of the two types of steel in the section. This conventional analysis typically underestimates the time-dependent change of concrete stress.

It should be noted that creep, shrinkage and relaxation change the distribution of stresses between the concrete, the prestressed steel and the nonprestressed steel, without change in the magnitude and location of the stress resultants. Thus, it can be verified that the stress distributions depicted in Figs. 4(a) and 4(b) and the forces in the steel at times t_o and t are statically equivalent to $\{N, M_x, M_y\}$ introduced at t_o. The compatibility of strain in concrete at any point and the reinforcement bonded to it can also be verified at t_o and t.

CONCLUDING REMARKS

An analysis procedure is presented for calculating the immediate and the long-term stresses and strains in prestressed or nonprestressed concrete sections of general shapes. Requirements of equilibrium and compatibility are satisfied. Equilibrium means that, for each of the two stages, the calculated stresses on the concrete and the reinforcements have resultants equal to the normal force and moments applied on the sections. To satisfy the compatibility requirement, the strains in the concrete and the reinforcement bonded to it are equal at all locations.

No empirical equation is used to estimate the loss in tension in the prestressed steel. A conventional analysis, which uses such an equation and applies the prestress force after loss on a plain concrete section to determine the long-term stresses, generally does not satisfy the compatibility requirement. Substantial overestimation of the long-term stresses and curvatures caused by prestressing can result, particularly when the nonprestressed steel ratio is high (more than one percent).

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APPENDIX A — NUMERICAL EVALUATION OF AREA PROPERTIES OF CROSS SECTIONS OF ARBITRARY SHAPES

A procedure is given below for evaluating an integral over an area of arbitrary shape [see Fig. A1(a)]. The integral is:

$$H_{mn} = \int x^m y^n \, dA \tag{A1}$$

with m and n being positive integers. Setting m and n equal to 0, 1 or 2, the integral will give the values of A, B_x, B_y, I_x, I_y or I_{xy}, defined by Eqs. (15) to (17).

A curved perimeter is to be idealized by a closed polygon of N sides [see Fig. A1(a)]. The geometry of the polygon is defined by the (x, y) coordinates of N nodes numbered in the sequence indicated in the figure.

The integral in Eq. (1) can be expressed as a summation:

$$H_{mn} = \sum_{i=1}^{N} \Delta_i \tag{A2}$$

where Δ_i is the value of the integral over the area between the x axis and the ith side of the perimeter [see Fig. A1(b)]. Introduce the variable:

$$\bar{x} = x - x_i \tag{A3}$$

The y coordinate of any point on the ith side may be expressed as:

$$(y)_{i\text{th side}} = y_i + \frac{\Delta y_i}{\Delta x_i}\bar{x} \tag{A4}$$

where

$$\Delta x_i = x_{i+1} - x_i; \; \Delta y_i = y_{i+1} - y_i \tag{A5}$$

The value of the integral in Eq. (A1) over the area between the x axis and the ith side of the perimeter is:

$$\Delta_i = \int_0^{\Delta x_i} \int_0^{y_i + \frac{\Delta y_i}{\Delta x_i}\bar{x}} (x_i + \bar{x})^m y^n \, dy \, d\bar{x} \tag{A6}$$

or

$$\Delta_i = \frac{1}{n+1} \int_0^{\Delta x_i} (x_i + \bar{x})^m \left(y_i + \frac{\Delta y_i}{\Delta x_i}\bar{x} \right)^{n+1} d\bar{x} \tag{A7}$$

Substitution of each of the two terms in Eq. (A7) by its binomial expansion and integration gives:

$$\Delta_i = \frac{1}{n+1} \left\{ \sum_{j=0}^{m} \left[c_{mj} x_i^{m-j} (\Delta x_i)^{j+1} \left(\sum_{k=0}^{n+1} c_{(n+1)k} \frac{y_i^{n+1-k} (\Delta y_i)^k}{k+j+1} \right) \right] \right\} \tag{A8}$$

where the binomial coefficient:

$$c_{mj} = \frac{m!}{j!(m-j)!} \tag{A9}$$

Substitution of the appropriate integers m and n in Eqs. (A7) and (A3), the area properties of the cross section in Fig. A1 can be expressed as:

$$A = H_{00} = \sum_{i=1}^{N} \left[\Delta x_i \left(y_i + \frac{\Delta y_i}{2} \right) \right] \tag{A10}$$

$$B_x = H_{01} = \sum_{i=1}^{N} \left[\frac{\Delta x_i}{2} \left(y_i^2 + y_i \Delta y_i + \frac{(\Delta y_i)^2}{3} \right) \right] \tag{A11}$$

$$B_y = H_{10} = \sum_{i=1}^{N} \left[x_i \Delta x_i \left(y_i + \frac{\Delta y_i}{2} \right) + (\Delta x_i)^2 \left(\frac{y_i}{2} + \frac{\Delta y_i}{3} \right) \right] \tag{A12}$$

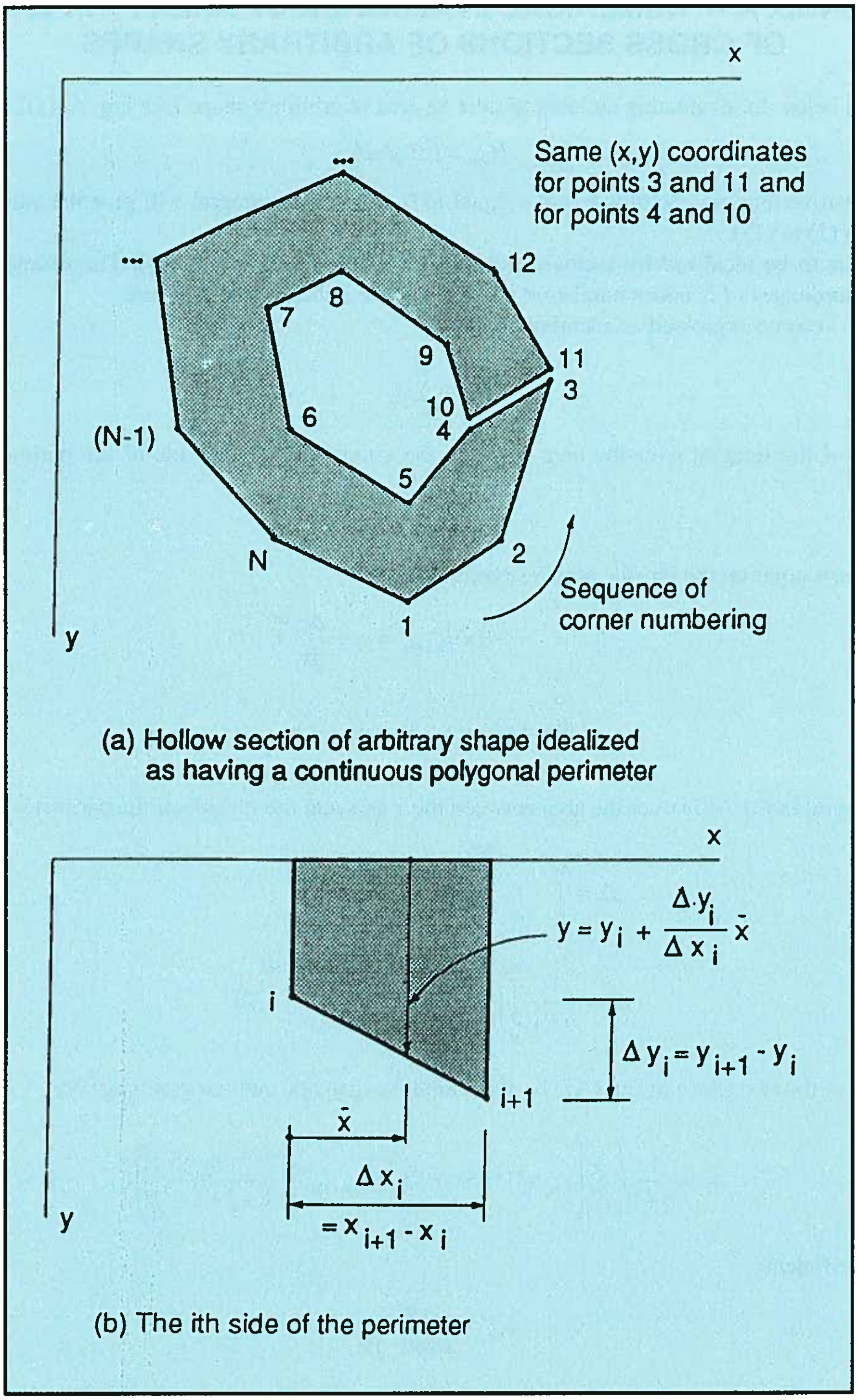

Fig. A1. Evaluation of area properties of a cross section of arbitrary shape.

$$I_x = H_{02} = \sum_{i=1}^{N}\left[\frac{\Delta x_i}{3}\left(y_i^3 + \frac{3y_i^2 \Delta y_i}{2} + y_i(\Delta y_i)^2 + \frac{(\Delta y_i)^3}{4}\right)\right] \quad \text{(A13)}$$

$$I_y = H_{20} = \sum_{i=1}^{N}\left[x_i^2 \Delta x_i\left(y_i + \frac{\Delta y_i}{2}\right) + 2x_i(\Delta x_i)^2\left(\frac{y_i}{2} + \frac{\Delta y_i}{3}\right) + (\Delta x_i)^3\left(\frac{y_i}{3} + \frac{\Delta y_i}{4}\right)\right] \quad \text{(A14)}$$

$$I_{xy} = H_{11} = \sum_{i=1}^{N}\left[\frac{x_i \Delta x_i}{2}\left(y_i^2 + y_i \Delta y_i + \frac{(\Delta y_i)^2}{3}\right) + \frac{(\Delta x_i)^2}{2}\left(\frac{y_i^2}{2} + \frac{2y_i \Delta y_i}{3} + \frac{(\Delta y_i)^2}{4}\right)\right] \quad \text{(A15)}$$

APPENDIX B — NOTATION

$A, B_x, B_y, I_x, I_y, I_{xy}$ = area of transformed section, its first and second moments about x and y axes

$\bar{A}, \bar{B}_x, \bar{B}_y, \bar{I}_x, \bar{I}_y, \bar{I}_{xy}$ = area properties of age-adjusted transformed section

a, b = intercepts of neutral axis with x and y axes

E_c = modulus of elasticity of concrete

$\bar{E}_c$ = age-adjusted modulus of elasticity of concrete

E_{ns}, E_{ps} = modulus of elasticity of nonprestressed and prestressed steel, respectively

M_x = moment about x axis

M_y = moment about y axis

N = normal force

ns, ps = subscripts referring to nonprestressed and prestressed steel, respectively

O = subscript indicating strain or stress at the reference Point O

t = instant at which time-dependent strain or stress is determined

t_o = instant at which forces and prestressing are introduced

ε = normal strain

ε_{cs} = free shrinkage of concrete (generally a negative value)

σ = normal stress

φ = creep coefficient

χ = aging coefficient

$\Delta\bar{\sigma}_{pr}$ = reduced relaxation of prestressed steel (generally a negative value)

γ_x, γ_y = slope of stress distribution ($= \partial\sigma/\partial y$ and $\partial\sigma/\partial x$, respectively)