Deformation Controlled Nonlinear Analysis of Prestressed Concrete Continuous Beams

A finite element model, based on a curvature increment technique, is applied to the nonlinear analysis of prestressed concrete continuous beams. The computer model is capable of predicting the response of such beams over the entire loading range from precracking to the collapse stage.

Nonlinearity is included in the model by means of a stepwise linear analysis technique, which makes use of the relevant secant stiffnesses obtained from the moment-curvature relationships for segments along the length of the beam. The use of curvature, as opposed to load, as the incrementing parameter facilitates the analysis of a beam with limited ductility, in which critical regions soften before failure occurs.

A parametric study carried out to determine the optimum layout of the segments for the model is described. The optimum mesh is used in the analysis of a number of beams and the results are compared with test data and other analytical predictions. It is concluded that the macroscopic finite element model gives an adequate prediction of the behavior of a continuous prestressed concrete beam over the entire range of loading up to failure.
crete Continuous Beams (NAPCCB), which has been developed utilizing a macroscopic finite element procedure, is outlined in this paper. The validity of the procedure is established by means of comparison with test data and other analytical predictions.

**THE COMPUTER PROGRAM**

The NAPCCB computer program is based on the stiffness method and was developed using the numerical procedure suggested and used by Warner and Yeo. The program uses a curvature incrementing technique and is capable of tracing the response of a bonded prestressed concrete beam of rectangular, I or T section, subjected to either concentrated or uniformly distributed loading, over the entire range of behavior from prestressing to failure. A flow chart of the computer program is given in Appendix A.

The beam is idealized in the analysis by dividing it into segments, where a segment is defined as the length between two consecutive joints, with each joint having two degrees of freedom, namely rotation and displacement. A linear elastic analysis is carried out to determine the effects of prestress and then the moment-curvature relationships for all the segments are established. A key segment, which is one located at a region where failure is likely to occur in the beam, is selected and the curvature in this segment is incremented in steps.

For each increment of curvature, a linear analysis, using the relevant secant bending stiffness for each segment, is carried out to determine the moment and curvature in each of the segments. At each increment of curvature, the cycle of calculation is repeated until one of the segments reaches its ultimate curvature capacity, at which stage failure of the beam is said to occur. Further details of the steps involved in the analytical procedure have been given by Warner and Yeo and Kodur.

The moment-curvature relationship is generated for the central section of a segment using the strain compatibility technique. This relationship is assumed to be valid throughout the length of the segment. For each segment the moment-curvature relationship is generated for both positive and negative bending, since some segments switch from positive to negative bending and vice-versa, due to redistribution of moment in the inelastic range. The ultimate moment capacity, $M_u$, of a section is taken as that corresponding to the ultimate curvature, $K_u$, which is the curvature corresponding to a concrete strain, $\varepsilon_{\text{max}}$, in the extreme compression fiber or to a steel strain equal to ultimate strain of the tension reinforcement at the section.

The two options in the model for selecting the stress-strain relationship for concrete in compression, one for unconfined concrete and the other for confined concrete, are shown in Fig. 1. The factors, $K_f$, and $Z_m$, used in Fig. 1(b), are defined as:

$$K_f = 1 + \frac{\rho_s f_y}{f'_c} \tag{1}$$
and

\[
Z_m = \left[ \frac{(3 + 0.29 f'_c)}{(145 f'_c - 1000)} \right]^{0.5} + 0.75 \rho_s \sqrt{\frac{h''}{S_h}} - 0.002 K_f
\]  \hspace{1cm} (2)

where

- \( \rho_s \) = ratio of volume of hoop reinforcement to volume of core measured to outside of the hoops
- \( f_{sh} \) = yield strength of the hoop reinforcement (MPa)
- \( f'_c \) = compressive cylinder strength of concrete (MPa)
- \( h'' \) = width of concrete core measured to outside of the peripheral hoops (mm)
- \( S_h \) = center-to-center spacing of hoop sets (mm)

The maximum usable compressive strain of concrete, \( \varepsilon_{\text{max}} \), depends on the confinement resulting from hoop reinforcement present in the section and is given by:

\[
\varepsilon_{\text{max}} = 0.004 + 0.9 \rho_s \left( \frac{f_{sh}}{300} \right)
\]  \hspace{1cm} (3)

For unconfined concrete, where \( \rho_s \) is zero, the maximum usable strain is 0.004.

For concrete in tension, the stress-strain relationship is assumed to be linear with a slope equal to that of the curve for compression at zero stress. The contribution of concrete in the tension zone, after cracking, is not taken into account. An elastic-plastic stress-strain relationship is used for nonprestressed reinforcement, both in compression and tension. A multilinear stress-strain relationship is used for prestressed reinforcement in tension.

As shown in Fig. 2, the moment-curvature relationship for a prestressed concrete section does not usually pass through an origin corresponding to zero moment and zero curvature. Selection of a suitable origin is an important step in the analysis since nonlinearity is accommodated by updating the secant stiffness, defined as the slope of the line from the origin to the relevant point on the moment-curvature diagram. The suitability of using the three possible origins suggested by Arenas,

namely O, D and D' in Fig. 2, for the moment-curvature relationship was investigated.

If Point O is selected as the origin, computational problems such as non-convergence are encountered since the secant stiffness is negative for applied moments in the range from zero to \( M_o \) and becomes infinite at moment \( M_o \). When Point D is selected as the origin, it is found that the effects of prestress are not included in the analysis, since this choice implies zero curvature and zero moment at zero applied load. However, selection of Point D' as the origin, where \( D' \) is defined by the magnitude, \( M_{sec} \), of the secondary moment at the section due to prestress, accounts for prestressing effects at all stages of loading and computational problems such as non-convergence are avoided. Hence, the moment-curvature relationship referred to the \( M' - K' \) axes is used in the analysis. This change of axis is a computational device only and the total moments are evaluated at each stage of loading.

The use of curvature as the incrementing parameter has certain advantages over the load incrementing technique.\(^7\) \(^8\) Curvature incrementing facilitates the analysis of beams where one or more critical sections undergo local softening (moment shedding) before failure. Also, the complete response of the beam can be determined in one computer run since the moment-curvature relationships are generated before the start of the nonlinear analysis, as opposed to several runs required in computer programs such as PCFRAME,\(^9\) based on a load incrementing technique.

### FAILURE LOAD OF A CONTINUOUS BEAM

Subsequent to cracking of the concrete, the bending moment diagram for a continuous prestressed concrete beam under load deviates from that given by a linear elastic analysis and, with increasing load, approaches that obtained from a plastic collapse anal-

![Fig. 2. Axes system for moment-curvature relationship.](image-url)
ysis. The failure load, based on a plastic collapse mechanism approach, can be attained only if the sections at which plastic hinges form have sufficient rotational capacity to allow full development of the required number of plastic hinges for formation of a mechanism in the beam. The rotational capacity available at a plastic hinge can be assessed from the moment-curvature relationship of the cross section of the beam and increases with ductility.

Determination of the required rotation at a hinge location necessitates an analysis of the beam under a load level corresponding to the plastic collapse load. If the required rotation at any of the plastic hinge locations is greater than that available, the beam will fail prior to reaching the plastic collapse load since the plastic moment capacity cannot be maintained at this location and the collapse mechanism cannot develop fully. Partial redistribution of moment occurs in this case as opposed to full redistribution which accompanies the development of a plastic collapse mechanism.

In computing the required rotation at a plastic hinge location in a structure, inelastic deformations must be considered. This consideration has stifled the application of plastic methods of design to concrete framed structures. As a result, North American Codes\textsuperscript{10,11} use a lower bound approach to a failure load by utilizing elastic bending moments, which may be adjusted to account for redistribution of moment, to determine the load at which the ultimate moment capacity is attained at a single section. The permitted amount of redistribution of moment is based on a measure of the ductility of a specific cross section in a member. However, it has been suggested that redistribution of moment should be based on member ductility rather than on sectional ductility.\textsuperscript{2}

Redistribution of moment in statically indeterminate structures can be studied using nonlinear methods of analysis. The NAPCCB computer program has been developed as part of a study of redistribution of moments in continuous prestressed concrete beams.\textsuperscript{1,2} This program computes the failure load of a beam by determining the load level at which the ultimate curvature is exceeded at a hinge location. Utilization of the program is incorporated in the analysis of the strength of a two-span beam presented in Appendix B.

**OPTIMUM IDEALIZATION OF BEAM**

Proper modeling of the regions in the beam where failure occurs is an important aspect in the NAPCCB computer model since the failure load is governed by the strength of such critical sections. Warner and Yeo\textsuperscript{2} suggested the use of an aspect ratio (ratio of length to depth of the segment) of 1 to 2.

In order to assess an optimum element size, two prestressed concrete beams, one simply supported and the other two-span continuous, were analyzed using different idealizations in which the aspect ratio was reduced gradually in the critical regions, as shown in Figs. 3 and 4, respectively. The aspect ratio of the segments in

Fig. 3. Idealized models for simply supported beam.
the critical regions for each idealization is given in Tables 1 and 2. In the case of the simply supported beam, the critical region is at the load point, while the load point and the central support are the critical regions in the continuous beam.

Both beams have a rectangular cross section, 250 mm (9.84 in.) deep and 150 mm (5.91 in.) wide, and the amounts of reinforcement and the material properties are as given in Figs. 3 and 4, respectively. The model designation, in Figs. 3 and 4 and in Tables 1 and 2, consists of two digits at the beginning identifying the number of segments in the beam, followed by letters SEG for SEGments and SB or CB for Simply supported or Continuous Beam.

Results from the analyses of the idealized simply supported and continuous beams are presented in Tables 1 and 2, respectively. From Table 1 it can be seen that the failure load predicted by NAPCCB for the simply supported beam is the same for all the idealizations and is equal to the plastic collapse load, based on the ultimate moment capacity at the critical section. This is expected since the plastic collapse load is determined by equilibrium considerations alone for a statically determinate structure. However, the deflection under the load point at the failure load decreases with the aspect ratio of the segment in the critical region.

The deflection of the beam depends on the area of the curvature diagram and, consequently, on the plastic hinge length, \( l_p \), as shown for the idealized curvature diagram\(^{13} \) in Fig. 5. The formulation in the model is such that \( l_p \) is equal to the length of the segment adjacent to the load point section since the entire critical segment is assumed to be fully plastic at failure. Thus, a model with a higher aspect ratio, and related larger plastic hinge length, will predict a larger deflection for the same value of ultimate curvature at the critical section.

### Table 1. Results from simply supported beam analysis.

<table>
<thead>
<tr>
<th>Model designation</th>
<th>Aspect ratio</th>
<th>Failure load (kN)</th>
<th>Load point curvature (x10(^6)/mm)</th>
<th>Load point deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>06SEGBSB</td>
<td>2.0</td>
<td>47.23</td>
<td>136.57</td>
<td>78.51</td>
</tr>
<tr>
<td>12SEGBSB</td>
<td>1.0</td>
<td>47.23</td>
<td>136.57</td>
<td>50.17</td>
</tr>
<tr>
<td>14SEGBSB</td>
<td>0.5</td>
<td>47.23</td>
<td>136.57</td>
<td>31.93</td>
</tr>
<tr>
<td>16SEGBSB</td>
<td>0.25</td>
<td>47.23</td>
<td>136.57</td>
<td>22.16</td>
</tr>
<tr>
<td>18SEGBSB</td>
<td>0.125</td>
<td>47.23</td>
<td>136.57</td>
<td>17.49</td>
</tr>
<tr>
<td>Plastic analysis</td>
<td>—</td>
<td>47.23</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Metric (SI) conversion factors: 1 kip = 4.448 kN; 1 in. = 25.4 mm.

### Table 2. Results from continuous beam analysis.

<table>
<thead>
<tr>
<th>Model designation</th>
<th>Aspect ratio</th>
<th>Failure load (kN)</th>
<th>Curvature under load (x10(^6)/mm)</th>
<th>Curvature at support (x10(^6)/mm)</th>
<th>Load point deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12SEGCB1</td>
<td>1.0</td>
<td>186.71</td>
<td>27.96</td>
<td>-55.09</td>
<td>10.53</td>
</tr>
<tr>
<td>12SEGCB2</td>
<td>0.75</td>
<td>182.29</td>
<td>16.70</td>
<td>-55.09</td>
<td>9.00</td>
</tr>
<tr>
<td>15SEGCB</td>
<td>0.50</td>
<td>158.71</td>
<td>12.23</td>
<td>-55.09</td>
<td>7.12</td>
</tr>
<tr>
<td>18SEGCB</td>
<td>0.25</td>
<td>131.55</td>
<td>8.66</td>
<td>-55.08</td>
<td>5.16</td>
</tr>
<tr>
<td>21SEGCB</td>
<td>0.125</td>
<td>119.20</td>
<td>7.23</td>
<td>-55.08</td>
<td>4.26</td>
</tr>
<tr>
<td>24SEGCB</td>
<td>0.0625</td>
<td>113.77</td>
<td>6.63</td>
<td>-55.08</td>
<td>3.85</td>
</tr>
<tr>
<td>Plastic analysis</td>
<td>—</td>
<td>186.28</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Metric (SI) conversion factors: 1 kip = 4.448 kN; 1 in. = 25.4 mm.
Table 2 and Fig. 6 show that, for the continuous beam, the failure load decreases with aspect ratio. From Table 2 it can be seen that, while the support segment reaches its ultimate capacity, as characterized by the constant value of curvature at the support for all the idealizations, the load point segment reaches a level of curvature dependent upon the aspect ratio. This can be seen also from the moment-curvature relationship in Fig. 7, where the numbers refer to the six different idealizations as listed. Fig. 6 shows that the failure load approaches a value of 108 kN (24.3 kips) at zero aspect ratio, a value which could be interpreted as the actual failure load of the beam. Since failure loads lower than that given by a plastic analysis were obtained from the NAPCCB analyses, it may be concluded that only partial redistribution of moment occurred in the beam at failure.

The variation in the magnitude of the failure load and the related deflection of the continuous beam for the different idealizations can be explained with reference to Fig. 8. At failure of the beam, the rotation, $\phi_u$, in the plastic hinge at the central support can be determined using the free hinge approach. The rotation at failure is given by the relation:

$$\phi_u = \phi_2 - \phi_1$$  \hspace{1cm} (4)

where $\phi_1$ and $\phi_2$, which are the rotations at the end of a simply supported span subjected to a bending moment, $M_u$ [Fig. 8 (c)], and a load, $W_u$ [Fig. 8 (d)], respectively, can be computed using the relationships:

$$\phi_1 = \frac{M_u L}{3EI} \quad \text{and} \quad \phi_2 = \frac{5W_u L^2}{81EI}$$  \hspace{1cm} (5)

where $EI$ is the flexural stiffness of the span and $L$ is the span length. The rotation, $\phi_u$, is also dependent on the product of plastic hinge length, $l_p$, and ultimate curvature, $K_u$, at the central support and can be written as:

$$\phi_u = \frac{5W_u L^2}{81EI} - \frac{M_u L}{3EI} = K_u l_p$$  \hspace{1cm} (6)

From Eq. (6), it can be seen that, for a specific moment capacity, $M_u$,
and related curvature, $K_p$, an increase in plastic hinge length, $l_p$, as a result of increased aspect ratio of the critical segment, leads to a higher value of the failure load, $W_u$. In addition, the resulting increase in the area of the curvature diagram, resulting from an increased plastic hinge length, leads to a higher rotation in the plastic hinge length and hence a larger deflection at the load point. Thus, an increase in the aspect ratio of the segments in the critical regions leads to a higher failure load and a larger deflection in the model NAPCCB.

From the results of analyses of a number of additional two-span and three-span beams under various loading conditions, it was found that an aspect ratio in the range of 0.2 to 0.4 in the critical regions gives a good estimate of the failure load and the deflection in a beam in which partial redistribution of moment occurs. Use of an aspect ratio less than 0.1 induces round-off errors and also involves higher computer costs.

An investigation of the effect of refinement of mesh in two-span beams where redistribution of moment is virtually complete at failure indicated that the aspect ratio has no significant effect on failure load. In this case significant plastic deformation will occur in the region of the load point as well as at the central support. Since the rotation, $\phi_u$, is related to the area of the curvature diagram, the influence of an increase in the plastic hinge length at the support (negative curvature) will be counteracted by a similar increase at the load point (positive moment).

Consequently, from Eq. (4), if $\phi_u$ and $\phi_1$ (which is dependent on $M_u$) are invariant, so will $\phi_2$ be invariant, with the result that $W_u$ will not be dependent on $l_p$. However, the deflection decreases with the aspect ratio as in the case of beams with partial redistribution of moment. This can be attributed to the fact that the deflection at the load point will be influenced primarily by plastic deformation in the load point region. Thus, an increased aspect ratio and related plastic hinge length will lead to increased deflection.

It is recommended that, since the
extent of redistribution of moment is not known before the analysis is carried out, an aspect ratio of 0.2 to 0.4 in the critical regions, together with an aspect ratio of 1 to 2 in other regions, should be used in all beams to obtain a reasonable estimate of both the failure load and the deflection.

EVALUATION OF THE MODEL

The validity of the computer model was further evaluated by comparing its predictions with test data and with other analytical predictions. The analyses were carried out using an aspect ratio of 0.25 for segments in the critical regions. Confinement present in the section was taken into account by using the relevant stress-strain relationship for concrete in compression.

Test data\(^\text{12, 14}\) in the form of load-deflection and load-moment plots, for a two-span beam (S4) and a three-span beam (3S2), are compared with the results from the NAPCCB computer model in Figs. 9 and 10, respectively. In the load-moment plots the positive moment corresponds to that at the load point, while the negative moment is that at the interior support. Good agreement is obtained between the test data and the predicted responses over the entire load history for both beams.

The failure loads from tests, NAPCCB analyses and plastic analyses for Beams S4 and 3S2 are given in Table 3. The failure loads predicted by the nonlinear analysis and by the plastic analysis are close, indicating that full redistribution of moment occurred in both beams. The slightly higher failure load for the three-span beam is due to the use of maximum moments, as opposed to ultimate moments, at the critical sections which exhibited softening moment-curvature relationships.

The test failure load is approximately 15 percent higher than the load predicted from plastic analysis for each beam. The higher failure load has been attributed to the presence of the prestressing duct acting as extra reinforcement and neglect of

### Table 3. Failure load in beams.

<table>
<thead>
<tr>
<th>Beam designation</th>
<th>Test data</th>
<th>NAPCCB analysis</th>
<th>Plastic analysis</th>
<th>PCFRAME analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>S4</td>
<td>73.70</td>
<td>63.52</td>
<td>63.25</td>
<td>—</td>
</tr>
<tr>
<td>3S2</td>
<td>94.23</td>
<td>80.41</td>
<td>83.50</td>
<td>—</td>
</tr>
<tr>
<td>Rectangular</td>
<td>—</td>
<td>131.57</td>
<td>186.28</td>
<td>136.02</td>
</tr>
<tr>
<td>1</td>
<td>—</td>
<td>68.65</td>
<td>68.70</td>
<td>71.80</td>
</tr>
</tbody>
</table>

Metric (SI) conversion factor: 1 kip = 4.448 kN.
strain hardening of the nonprestressed steel, two effects which are not accounted for in the analysis. The smaller deflections predicted by NAPCCB at failure, as shown in Fig. 9, are due to the lower predicted failure loads.

Predictions from the NAPCCB computer model were also compared with results from a microscopic finite element model, PCFRAME, for two two-span beams, one of rectangular and the other of I cross section. The use of similar stress-strain curves for concrete and steel in both programs facilitate a direct comparison of results. An aspect ratio of 0.25 was chosen for the mesh size in the critical regions for both analyses. Since confinement of concrete is not accounted for in PCFRAME, the analyses were carried out using the unconfined stress-strain curve for concrete in compression. Good agreement between the results from two programs over the entire loading range can be seen from the load-deflection and the load-moment plots in Figs. 11 and 12, respectively.

In the case of the rectangular beam, as can be seen in Fig. 12(a), the support segment reached its ultimate capacity of -50.67 kN·m (-37.37 kip-ft), but the load point segment did not reach its ultimate capacity of +98.71 kN·m (+72.79 kip-ft), indicating that the degree of redistribution of moment was partial in this beam at failure. This is reflected in the large difference between the failure load from the plastic analysis and the NAPCCB analysis [Table 3 and Fig. 11(a)]. The failure load predicted by PCFRAME for this beam is slightly higher than that from NAPCCB because PCFRAME overestimates the strength of the critical sections.

It has been shown that this overestimation is due to the assumption in PCFRAME that the strain and the related stress is constant over the depth of a layer in an element. For the I beam, the predicted failure load from the NAPCCB analysis is very close to that from the plastic analysis, indicating that the redistribution of moment is almost complete. Again, it can be seen in Table 3 that PCFRAME overestimates the failure load. The slightly larger deflections at failure predicted by PCFRAME for both beams, as can be seen in Fig. 11, is due to the higher predicted failure loads.

As a result of the above evaluation, it is felt that the nonlinear mathematical model, NAPCCB, may be used with confidence to examine the effects of the various parameters which may influence the redistribution of moments in prestressed concrete continuous beams. Such a study is being undertaken in order to refine and extend the approach proposed by Mouessian and Campbell for the determination of the failure load of a continuous prestressed concrete beam.

**CONCLUSIONS**

1. An aspect ratio in the range of 0.2 to 0.4 for the segments in the critical regions should be used in order to yield acceptable results from the nonlinear analysis.
2. Predicted failure loads are sensitive to the aspect ratio of the segments at the critical regions in beams where incomplete redistribution of moment occurs at failure, but not in beams where redistribution of moment is complete at failure. Predicted deflections are sensitive to aspect ratio in all cases.

3. Comparison of results from the model contained in the computer program, NAPCCB, with results from laboratory tests and from other analytical predictions showed that the model is capable of predicting the behavior of a prestressed concrete continuous beam at all stages of loading.

REFERENCES


APPENDIX A — FLOW CHART FOR NAPCCB PROGRAM

START

INPUT DATA

Analyse for prestressing effects

Generate M-K relationship

Choose a Target Curvature (TC) in Key Segment (KS)

CYCLE = 0

CYCLE = CYCLE + 1

Set E(J) = NEI(J)

Unit Load Analysis - ULM

ULC(KS) = \frac{ULM(KS)}{E(KS)}

SF = \frac{TC}{ULC(KS)}

ULC(J) = \frac{ULM(J)}{E(J)}

K(J) = ULC(J) \times SF(J)

Find BM(J) for K(J) from M-K relationship

NEI(J) = \frac{BM(J)}{K(J)}

Is E(J) = NEI(J)

NO

YES

Solve for W

Find reactions and deflections

Is TC > K_u (J)

YES

Increment TC

STOP

APPENDIX A — FLOW CHART FOR NAPCCB PROGRAM


11. ACI Committee 318, “Building Code Requirement for Reinforced Concrete (ACI 318-89) and Commentary (ACI 318R-89),” American Concrete Institute, Detroit, MI, 1989.


Fig. B1 shows a two-span continuous fully prestressed concrete beam of T cross section. The beam is subjected to uniformly distributed load and has an idealized parabolic profile of the prestressing tendon. The prestressing force is assumed to be constant over the length of the beam. This beam has been analyzed previously by Lin and Thornton (PCI JOURNAL, V. 17, No. 1, January-February 1972, pp. 9-20).

A linear elastic analysis of the beam gives the secondary moment at the central support, due to prestress, as 1897 kN·m (1399 kip-ft).

The beam is symmetric about the central support and has critical sections, namely, the locations of potential plastic hinges, one at the support and one in each span. For a uniformly distributed load, the critical section in the span will be at a distance of approximately 0.4L from the end support. A distance of 0.3961L, which corresponds to the location of the central section of a segment in the NAPCCB analysis, has been assumed here.

Properties of the two critical sections, obtained using the strain compatibility approach, are presented in Table B1.

### Plastic Collapse Load

Assuming a collapse mechanism to form in each span (full redistribution of moment), where the plastic hinge in the span forms at a distance of 0.3961L from the end support, the plastic collapse load, \( w_{pl} \), is given by:

\[
w_{pl} = \frac{2[M_c L - M_{c, a}]}{a (L - a)L}
\]

Using this value of \( M_{c, a} \), the failure load, \( w_{f, e} \), based on elastic analysis, is given by:

\[
M = \frac{w L^2}{8}
\]

### Failure Load Based on Elastic Analysis

Assuming no redistribution of moment, the failure load, \( w_{f, e} \), based on elastic analysis, can be computed according to the CAN3-A23.3-M.84 and ACI 318-89/318R-89 Codes by determining the load at which one of the critical sections reaches its ultimate capacity.

Elastic analysis gives the moment at the central support as:

\[
m = -\frac{w L^2}{8}
\]

Using this value of \( M \) in Eq. (B2), the failure load, \( w_{f, e} \), based on elastic analysis, is given by:

\[
w_{f, e} = 74.58 \text{ kN/m (5.10 kips/ft)}
\]

---

**Table B1. Properties of critical sections.**

<table>
<thead>
<tr>
<th>Property</th>
<th>Span critical section</th>
<th>Support critical section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment ( (M_{u}) )</td>
<td>4765 kNm (3513 kip-ft)</td>
<td>3646 kNm (2688 kip-ft)</td>
</tr>
<tr>
<td>Curvature ( (\kappa_{u}) )</td>
<td>37.21 \times 10^{-6}/\text{mm} (945 \times 10^{-6}/\text{in.})</td>
<td>8.05 \times 10^{-6}/\text{mm} (204 \times 10^{-6}/\text{in.})</td>
</tr>
<tr>
<td>( x/d_p )</td>
<td>0.049</td>
<td>0.455</td>
</tr>
<tr>
<td>( M_{sec} )</td>
<td>571 kNm (554 kip-ft)</td>
<td>1897 kNm (1399 kip-ft)</td>
</tr>
</tbody>
</table>

Metric (SI) conversion factors: 1 in. = 25.4 mm; 1 kip-ft = 1.356 kNm.
Applying a similar procedure to the section at a distance 0.3961L from the end support gives a failure load, $w$, equal to 96.44 kN/m (6.59 kips/ft).

Since the analysis based on the support section gives the lower load, the failure load, $w$, is taken as 74.58 kN/m (5.10 kips/ft).

**Failure Load from NAPCCB**

The beam was analyzed by means of the program NAPCCB using the idealization based on the recommendations given in this paper. This analysis indicated failure of the beam by crushing of the concrete at the central support at a load, $w_f$, of 76.79 kN/m (5.25 kips/ft).

As indicated in the moment-curvature relationships for the two sections (Fig. B2), at failure, the central support section reached its ultimate capacity, while the critical span section reached a level of moment (and curvature) well below the ultimate capacity.

**Failure Load Incorporating Redistribution of Moment**

(a) CAN3-A23.3-M84

This Code allows a percentage of redistribution of moment, for $(x/d_c)$ values in the range 0.2 to 0.6, of $[30 - 50 (x/d_c)]$ with an upper limit of 20 percent.

Hence, for the support section, where $x/d_c = 0.455$, the redistribution of moment is 7.25 percent. The central support section will attain its moment capacity, $M_c$, when:

$$M = -\frac{wL^2}{8}(1 - 0.0725) = M_c - M_{sec}$$

$$= -3646 - 1897 = -5543 \text{ kN.m}$$

giving:

$$w = w_{\text{CAN}} = 80.44 \text{ kN/m}$$

(5.51 kips/ft)

The moment, allowing for redistribution of support moment, at the span critical section due to this load is 4271 kN.m (3150 kip-ft), which is less than the moment capacity (4765 kN.m) of the section. Since $w_{\text{CAN}}$ is higher than $w_f$, this Code overestimates the load carrying capacity of the beam.

(b) ACI 318-89/318R-89

This Code allows a percentage of redistribution of moment of:

$$[\rho P + (\omega - \omega')]$$

$$= 20 \left[ \frac{\omega_i + \frac{d}{d_p} (\omega - \omega')}{0.36 \beta_i} \right]$$

Application of Eq. (B4) to the central support section results in a negative value, indicating that redistribution of moment is not permitted in this beam. Hence, the failure load according to this Code is the same as that computed previously neglecting redistribution of moment:

$$w_{\text{ACI}} = w_{le} = 74.58 \text{ kN/m}$$

(5.10 kips/ft)

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APPENDIX C — NOTATION

\( A_{ps} \) = area of prestressed reinforcement  
\( a \) = distance from end support to span critical section  
BM = bending moment in a segment  
\( d \) = distance from extreme compression fiber to centroid of nonprestressed reinforcement  
\( d_c \) = distance from extreme compression fiber to centroid of tension reinforcement  
\( d_p \) = distance from extreme compression fiber to centroid of prestressed reinforcement  
\( E_c \) = modulus of elasticity of concrete  
\( E_{ps} \) = modulus of elasticity of prestressed reinforcement  
\( E_s \) = modulus of elasticity of nonprestressed reinforcement  
e = eccentricity of prestressing tendon  
\( EI \) = flexural rigidity  
\( f_c \) = longitudinal stress in concrete at strain \( \varepsilon_c \)  
\( f'_c \) = compressive cylinder strength of concrete  
\( f_{pu} \) = ultimate stress in prestressed reinforcement  
\( f_{se} \) = effective stress in prestressed reinforcement  
\( f_r \) = modulus of rupture of concrete  
\( f_y \) = yield stress of nonprestressed reinforcement  
\( f_{sh} \) = yield stress of hoop reinforcement  
\( h' \) = width of concrete core measured to outside of the peripheral hoops  
\( K \) = curvature  
\( K_f \) = factor defined by Eq. (1)  
\( K_o \) = initial curvature due to prestress  
\( K_u \) = curvature at ultimate  
\( K_y \) = curvature at yielding of steel  
\( L \) = length of a span  
\( I_p \) = plastic hinge length  
\( M \) = moment  
\( M_c \) = ultimate moment of resistance at central support section  
\( M_{cr} \) = cracking moment  
\( M_s \) = ultimate moment of resistance at span critical section  
\( M_{sec} \) = secondary moment  
\( M_n \) = ultimate moment of resistance  
\( M_y \) = moment at yielding of steel  
\( NEI \) = new flexural rigidity  
\( S_h \) = center-to-center spacing of hoop sets  
SF = scaling factor  
ULC = curvature corresponding to ULM  
ULM = moment in a segment due to unit load intensity  
\( W \) = concentrated load  
\( W_u \) = failure load  
\( \varphi \) = uniformly distributed load  
x = distance from extreme compression fiber to neutral axis  
\( Z_{em} \) = factor defined by Eq. (2)  
\( \varepsilon_{c} \) = longitudinal strain in concrete  
\( \varepsilon_{max} \) = maximum usable compressive strain in concrete  
\( \phi_{u} \) = rotation at failure  
\( \phi_{1} \) = rotation in a simply supported beam with moment \( M_u \) at one end  
\( \phi_{2} \) = rotation due to load \( W_u \) in a simply supported beam  
\( \rho_{s} \) = ratio of volume of hoop reinforcement to volume of core measured to outside of hoops  
\( \beta_{1} \) = factor defined in Section 10.2.7.3 of ACI 318-89/318R-89  
\( \omega \) = reinforcement index of nonprestressed tensile reinforcement  
\( \omega' \) = reinforcement index of nonprestressed compressive reinforcement  
\( \omega_{p} \) = reinforcement index of prestressed reinforcement