



Shear and Torsion Design of Prestressed and Non-Prestressed Concrete Beams



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Design procedures which are based on rational models rather than empirical equations enable the engineer to develop a better understanding of actual structural behavior. In this regard, the unsatisfactory nature of current shear and torsion design procedures is evident if the ACI Code¹ chapter on shear and torsion is compared with the ACI chapter on flexure and axial load. In the flexure and axial load chapter a rational, simple, general method is explained in a few paragraphs of text.

On the other hand, the shear and torsion chapter consists of a collection of complex, restrictive, empirical equations which, while leading to safe designs, lacks an understandable central philosophy. This lack, in the opinion of the authors, is the source of many of the complaints which arise from the engineering profession about modern design codes becoming unworkably complicated.

In this paper an attempt is made to present procedures based on rational models which enable members containing web rein-

Shear and torsion design recommendations which are believed to be more rational and more general than current code provisions are presented. The use of the design recommendations is illustrated by means of several design examples. Comparisons with the results of other design methods are made.

forcement to be designed to resist shear and/or torsion.

In order to illustrate the characteristics of a rational model of structural behavior, the paper first briefly reviews the theory for flexure and axial load. Then the progress made in developing comparable rational models for torsion and shear is summarized. The way in which these models can be used to design prestressed and non-prestressed concrete beams for torsion and shear is explained.

In addition, design procedures for combinations of flexure and shear and flexure combined with shear and torsion are presented. Minimum reinforcement requirements, diagonal crack control requirements and detailing requirements are also discussed. Finally, the recommended procedures are summarized in a set of specific design recommendations, the use of which are illustrated by means of several design examples. Derivations of the major equations presented are included in three appendices at the end of the paper.

Plane Sections Theory for Flexure and Axial Load

The "plane sections" theory which is capable of predicting the response of prestressed and non-prestressed concrete beams loaded in flexure and axial load is described in several textbooks (e.g., Refs. 2, 3 and 4). This theory will be briefly illustrated here in order to review concepts which will be used in developing the models for torsion and shear.

Assume that it is desired to find the moment-curvature relationship for the rectangular prestressed concrete beam shown in Fig. 1. Since it is assumed that plane sections remain plane only two variables (say the concrete strain at the top, and the depth to the neutral axis) are required to define the concrete longitudinal strain distribution. For a chosen value of top concrete strain, a trial value of the depth of compression can be selected and the concrete strain distribution will then be fixed.

The longitudinal concrete stresses can be found from the concrete strains by using the concrete stress-strain characteristics. Usually, it is assumed

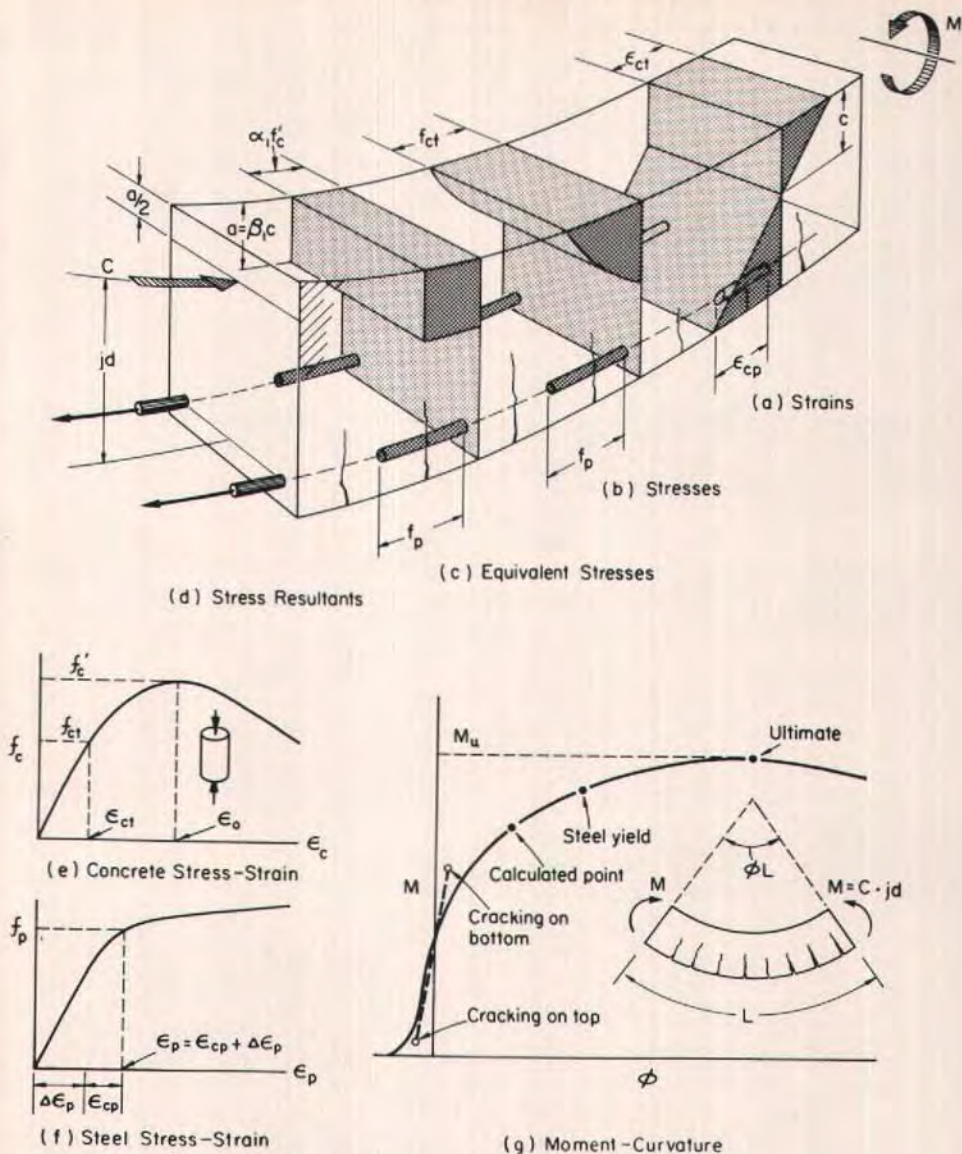


Fig. 1. Plane sections theory for flexure showing various relationships.

that in compression the stress-strain curve obtained from a test cylinder can be used and that in tension the concrete is not capable of resisting stress after cracking.

Due to the prestressing operation the strain in the prestressing steel will be substantially greater than the strain

in the surrounding concrete. For example, for a pretensioned beam before release the concrete strain is zero while the prestressing steel has a high tensile strain. This difference in strain, $\Delta\epsilon_p$, which is caused by and can be calculated from the specifics of the prestressing operation, is assumed

to remain constant throughout the life of the beam. For the concrete strain distribution being investigated, the strain in the concrete surrounding the prestressing steel is known and hence by adding the strain difference, $\Delta\epsilon_p$, the total strain in the prestressing steel, ϵ_p , can be determined. From the stress-strain characteristics of the prestressing steel, the stress, f_p , corresponding to the strain, ϵ_p , can be determined.

Knowing the stresses acting on the cross section, the resulting compressive force in the concrete and the tensile force in the steel can be computed. In the case of zero axial load, equilibrium requires that the compressive force in the concrete equals the tensile force in the steel. If this condition is not satisfied, the trial value of the depth of compression must be adjusted and the calculations repeated.

When the correct value of the depth of compression has been found, the moment corresponding to the chosen value of top concrete strain can then be calculated. This moment along with the curvature calculated from the strain distribution, will give one point on the moment-curvature plot. Repeating the calculations for different values of top concrete strain will produce the complete moment-curvature relationship shown in Fig. 1.

The moment-curvature relationship predicted on the basis that the concrete cannot resist tensile stresses is shown by the solid line in Fig. 1(g). The dashed line in Fig. 1(g) indicates the predicted precracking response if tensile stresses in the concrete are accounted for. Also shown are the cracking loads for the beam which will of course depend on the tensile strength of the concrete. Since this member is eccentrically prestressed, the concrete on the top face will crack if the applied moment is too low.

In determining the magnitude and

position of the resultant compression in the concrete, it is convenient to replace the actual stress distribution with an equivalent uniform stress distribution. Thus, the distribution shown in Fig. 1(b) could be replaced by a uniform stress of $\alpha_1 f'_c$ acting over a depth $\beta_1 c$, Fig. 1(c), where the stress block factors α_1 and β_1 have been chosen so that the magnitude and position of the resultant compression do not change. For a constant width of beam, the value of α_1 and β_1 will depend only on the shape of the concrete stress-strain curve, and the value of the highest concrete strain. The way in which these factors may be evaluated for a particular concrete stress-strain curve is shown in Appendix A.

In the ACI Code¹ the plane sections theory is the basis for determining the moment capacity. For this determination the following additional assumptions are made:

- (a) The maximum moment will occur when the compressive strain at the extreme fiber is 0.003.
- (b) The value of the stress block factor α_1 is 0.85.
- (c) The value of the stress block factor β_1 is 0.85 for concrete strengths of 4000 psi or less and is reduced continuously by 0.05 for each 1000 psi of strength in excess of 4000 psi but β_1 shall not be taken less than 0.65.*

These assumptions, of course, apply to both prestressed and non-prestressed members. In addition, the ACI Code¹ permits the use of an approximate expression for the stress in the prestressing steel at ultimate in lieu of a more accurate determination based on strain compatibility.

*For SI units β_1 shall be taken as 0.85 for strengths f'_c up to 30 MPa and shall be reduced continuously at a rate of 0.08 for each 10 MPa of strength in excess of 30 MPa but β_1 shall not be taken less than 0.65.

Truss Models for Shear and Torsion

Early design procedures for reinforced concrete members in shear were based on the truss analogy developed at the turn of the century by Ritter⁵ (1899) and by Mörsch⁶ (1902). This theory, which assumes that concrete is not capable of resisting tension, postulates that a cracked reinforced concrete beam (see Fig. 2) acts as a truss with parallel longitudinal chords and with a web composed of diagonal concrete struts and transverse steel ties. When shear is applied to this truss, the diagonal struts go into compression while tension is produced in the transverse ties and in the longitudinal chords.

Examination of the free body diagram of Fig. 2(b) reveals that the shear, V , is resisted by the vertical component of the compression force, D , in the diagonal struts. The horizontal component of the compression in the struts must be balanced by tension in the longitudinal steel. The magnitude of this tension will thus be given by:

$$\Delta N = \frac{V}{\tan\theta} \quad (1)$$

where θ is the angle of inclination of the diagonal struts. It can be seen from Fig. 2(c) that the diagonal compressive stress, f_d , is given by:

$$f_d = \frac{V}{b_v d_v \sin\theta \cos\theta} \quad (2)$$

where b_v is the effective web width and d_v is the effective shear depth.

Examination of the free body diagram of Fig. 2(d) shows that the tension in a transverse tie is given by:

$$A_v f_v = \frac{V_s}{d_v} \tan\theta \quad (3)$$

In discussing the choice of the angle of inclination of the diagonals, θ , Mörsch⁷ in 1922 made the following statement:

"We have to comment with regards to practical application that it is absolutely impossible to mathematically determine the slope of the secondary inclined cracks according to which one can design the stirrups. For practical purposes one has to make a possibly unfavorable assumption for the slope θ and therefore, with $\tan 2\theta = \infty$, we arrive at our usual calculation for stirrups which presumes $\theta = 45$ deg. Originally this was derived from the initial shear cracks which actually exhibit this slope."

The equation for the amount of transverse reinforcement needed which resulted from Mörsch's assumption that θ equals 45 deg became identified as the truss equation for shear.

Experience with the 45-deg truss analogy revealed that the results of this theory were typically quite conservative, particularly for beams with small amounts of web reinforcement. As a consequence, in North America it became accepted design practice to add an empirical correction term to the 45-deg truss equations. In the ACI Code this added shear capacity is taken as equal to the shear at the commencement of diagonal cracking and is often termed the "concrete contribution." As prestressing increases the diagonal cracking load, the beneficial effects of prestress are accounted for in the ACI Code by increasing the "concrete contribution."

The truss analogy predicts that in order to resist shear a beam needs both stirrups and longitudinal steel. The ACI Code,¹ rather than specifying the amount of additional longitudinal steel required for shear, gives rules for the extension of the flexural reinforcement (e.g., "reinforcement shall extend beyond the point at which it is

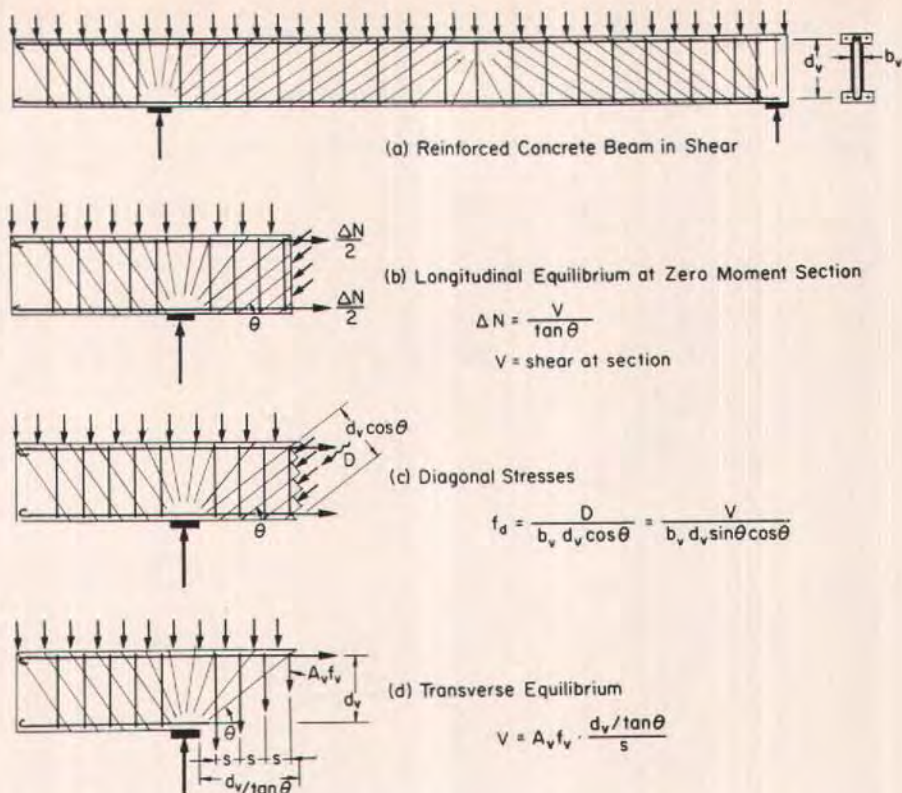


Fig. 2. Truss model for shear showing various relationships.

no longer required to resist flexure for a distance equal to the effective depth of the member").

The recent CEB Code⁸ has recognized that the angle of inclination of the concrete struts is not in general 45 deg. This code permits $\tan \theta$ to be varied between 3/5 and 5/3. These limits are a modification of the empirical limits determined by Lampert and Thürlimann⁹ for beams in torsion. It can be seen from Eqs. (1) and (3) that if a lower angle of inclination of the diagonal struts is chosen, then less transverse reinforcement but more longitudinal reinforcement will be required. Even though the CEB Code allows a range of values for θ , it still finds it necessary to include an empirical correction term (a "concrete

contribution") for lightly reinforced members.

The truss analogy equations for torsion were first developed by Rausch¹⁰ in 1929. As in shear, it is assumed that after cracking the concrete can carry no tension and that the beam acts as a truss with longitudinal chords and with walls composed of diagonal concrete struts and transverse steel ties (see Fig. 3).

Fig. 3 illustrates that the torsion is resisted by the tangential components of the diagonal compression which produce a shear flow, q , around the perimeter. This shear flow is related to the applied torque by the equilibrium equation:

$$T = 2A_o q \quad (4)$$

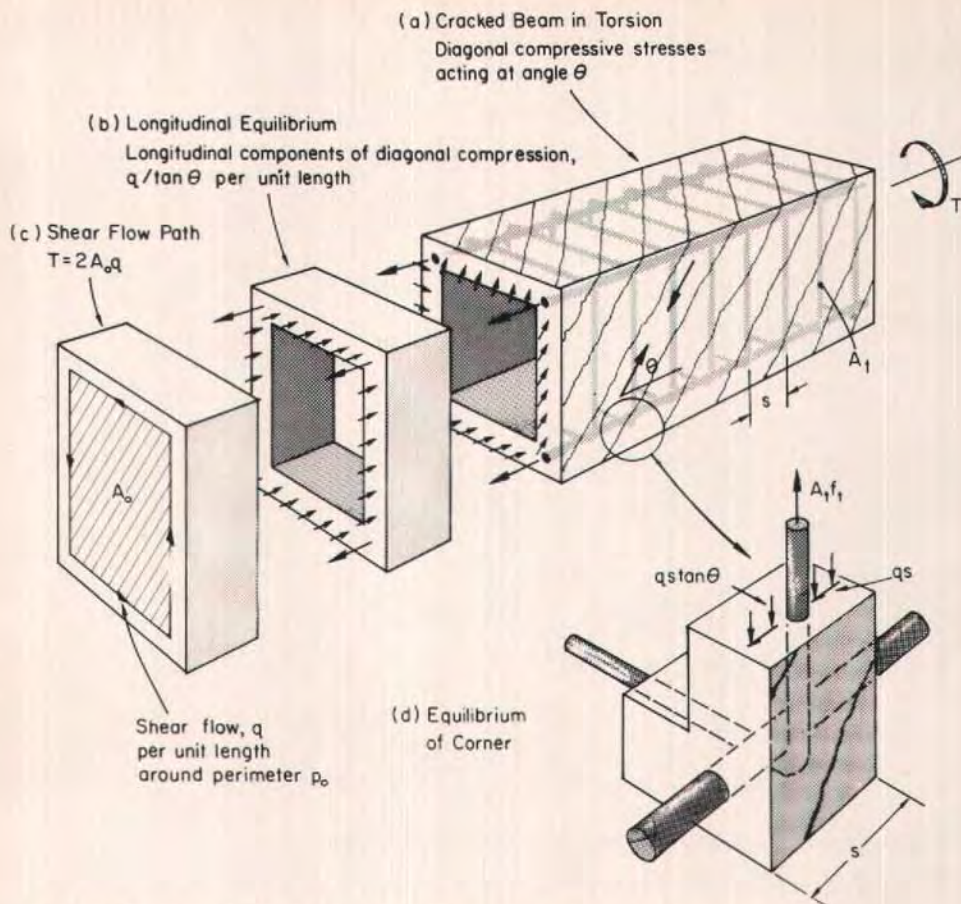


Fig. 3. Truss model for torsion showing assumed forces acting on element.

where A_o is the area enclosed by the shear flow path.

The longitudinal component of the diagonal compression must be balanced by tension in the longitudinal steel [see Fig. 3(b)], given by:

$$\Delta N = q \frac{p_o}{\tan \theta} = \frac{T}{2A_o} \frac{p_o}{\tan \theta} \quad (5)$$

To balance out the horizontal compression in the concrete, the resultant tension force in the steel must act at the centroid of the perimeter p_o .

An examination of the equilibrium of a corner element, shown in Fig.

3(d), indicates that the force in each hoop is:

$$A_s f_s = s q \tan \theta = \frac{T s}{2A_o} \tan \theta \quad (6)$$

Rausch, like Mörsch, assumed θ to be 45 deg. In addition, he assumed that the path of the shear flow coincided with the centerline of the closed stirrups. The resulting equations became identified as the truss equations for torsion.

In the ACI Code¹ the expressions for torsional strength consist of a modified form of the 45-deg truss

equations. These modifications, primarily based on the work of Hsu¹¹ and Mattock,¹² consist of adding an empirical "concrete contribution" related to the diagonal cracking load and replacing the "2" in Eqs. (5) and (6) by an empirical coefficient which is a function of the shape of the beam. While the ACI Code provisions do not treat prestressed concrete members in torsion, the recent *PCI Design Handbook*¹³ includes a torsion design procedure for prestressed concrete which is an extension of the ACI provisions. This procedure is based primarily on the work of Zia and McGee.¹⁴

The CEB Code⁸ recognizes that for torsion the angle of inclination of the diagonal struts is not always 45 deg. Again, this code permits $\tan\theta$ to vary between 3/5 and 5/3. In addition, rather than using the centerline of the closed stirrups as the shear flow path, the CEB Code, based on the work of Lampert and Thürlimann,⁹ uses a path defined by a line connecting the centers of the longitudinal bars in the corners of the closed stirrups.

Comparisons between the amounts of shear and torsion reinforcement required by the ACI and CEB Codes, and the authors' recommendations will be given later in this paper.

Compression Field Theory for Shear and Torsion

Before the equilibrium equations of the truss analogy can be used to design a member for shear and/or torsion, the inclination of the diagonal compression struts must be known. In 1929, Wagner¹⁵ dealt with an analogous problem in studying the post-buckling shear resistance of thin-webbed metal girders. He assumed that after buckling the thin webs would not resist compression and that the shear would be carried by a field of diagonal tension.

To determine the angle of inclination of the diagonal tension, Wagner considered the deformations of the system. He assumed that the angle of inclination of the diagonal tensile stress would coincide with the angle of inclination of the principal tensile strain. This approach became known as the tension field theory.

Applying Wagner's approach to reinforced concrete where it is assumed that after cracking the concrete can carry no tension and that the shear is carried by a field of diagonal compression results in the following expression for the angle of inclination of the diagonal compression:

$$\tan^2\theta = \frac{\epsilon_l + \epsilon_d}{\epsilon_t + \epsilon_d} \quad (7)$$

where

ϵ_l = longitudinal tensile strain

ϵ_t = transverse tensile strain

ϵ_d = diagonal compressive strain

This geometric equation can be thought of as a compatibility relationship which links the strains in the concrete diagonals, the longitudinal steel and the transverse steel.

Using the compatibility condition of Eq. (7), the equilibrium equations of the truss, and the stress-strain relationships of the concrete and the steel, the full behavioral response of reinforced concrete members in shear or torsion can be predicted. This approach is called the compression field theory.^{16,17}

To demonstrate how the compression field theory can be used to predict response, imagine that we wish to determine the behavior of a given beam subjected to a certain magnitude of shear. The solution could commence by assuming a trial value of θ . Knowing θ , the tensile stresses in the longitudinal and transverse steel and the diagonal compressive stresses in the concrete can be determined from

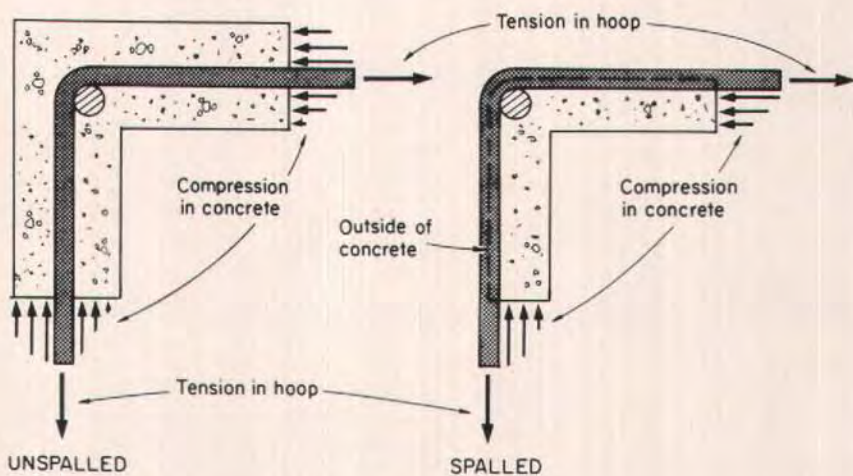


Fig. 4. Spalling of the concrete cover due to torsion.

the truss equilibrium relationships. Knowing the stress-strain characteristics of the reinforcement and the stresses in the reinforcement, the strains ϵ_t and ϵ_r can be determined.

Similarly, knowing the stress-strain characteristics of the concrete and the stress in the concrete, the strain, ϵ_d , can be determined. The calculated values of the strains can then be used to check the initial assumption of the

angle of inclination of the diagonal compression, θ . If the angle calculated agrees with the estimated angle, then the solution would be correct. If it does not agree, then a new estimate of θ could be made and the procedure repeated.

Thus, it can be seen that the compression field theory can predict the angle of inclination of the diagonal compression.

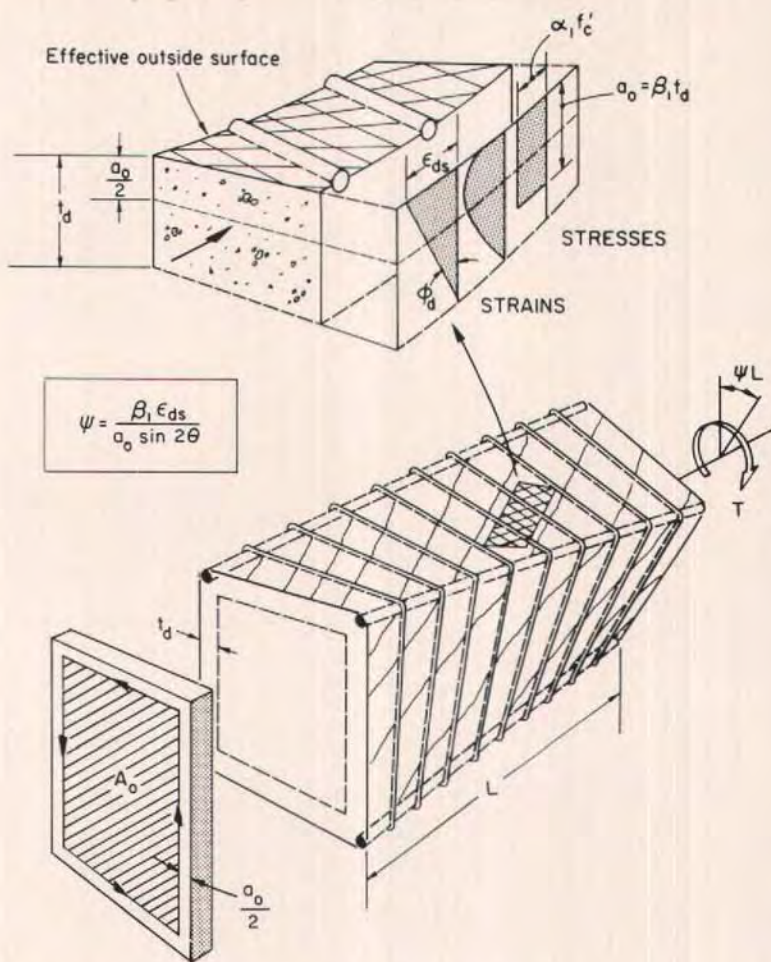


Fig. 5. Effective wall thickness of a twisted beam.

Members in Torsion

In applying the compression field theory to members in torsion, a few additional aspects of the behavior must be taken into account. In resisting the torsion, not all of the concrete is effective in providing diagonal compressive stresses. If the equilibrium of a corner element for a beam in torsion (Fig. 4) is examined, it can be seen that the compression in the concrete tends to push off the corner while the tension in the hoops holds it

on. Since concrete is weak in tension at higher torsions, the concrete outside of the hoops spalls off. Because of this spalling it is assumed that the effective outer surface of the concrete coincides with the hoop centerline.

If the deformed shape of the twisted beam in Fig. 5 is examined, it can be observed that the walls of the beam do not remain plane surfaces. Because of the curvature of the walls, the diagonal compressive strains will be a maximum, ϵ_{ds} , at the surface and will

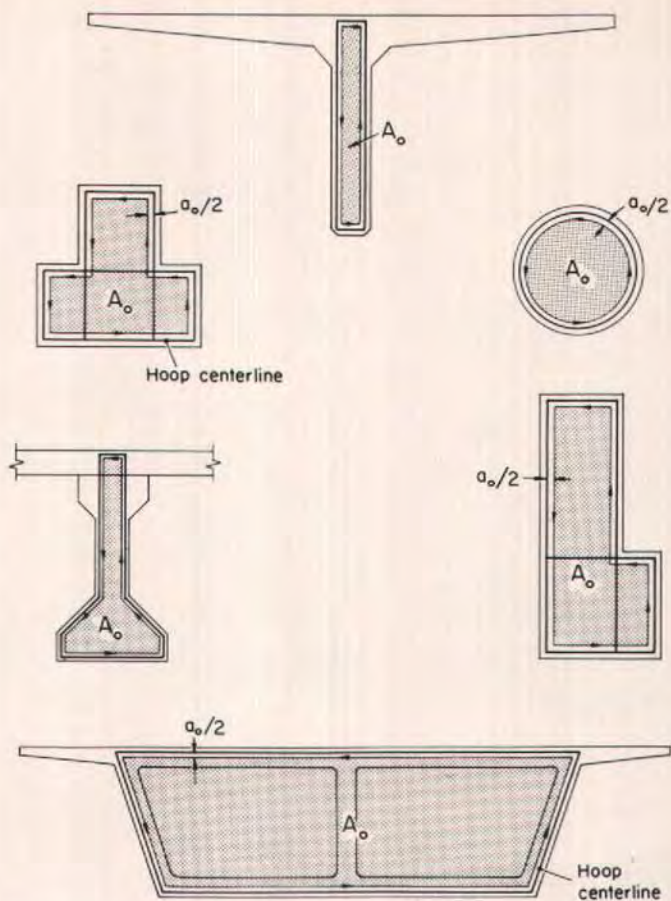


Fig. 6. Area enclosed by the shear flow for different member cross sections.

decrease linearly with the distance from the surface becoming tensile for depths below a certain distance, t_d . Thus, in torsion as in flexure, we have a depth of compression below which we may assume that the concrete, being in tension, is ineffective. The outside concrete spalls off and the inside concrete goes into tension; hence, we are left with a tube of effective concrete t_d thick which lies just inside the hoop centerline.

The diagonal concrete stresses will vary in magnitude over the thickness of the effective concrete tube from zero at the inside to a value f_{ds} corre-

sponding to the strain ϵ_{ds} at the effective outside surface. As in flexure we can replace this actual stress distribution by a uniform stress of $\alpha_1 f'_c = f_d$ acting over a depth of $\beta_1 t_d = a_o$ where the stress block factors α_1 and β_1 depend on the shape of the concrete stress-strain curve and the value of surface compressive strain, ϵ_{ds} . The centerline dimensions of the resulting tube of uniformly stressed concrete of thickness, a_o , will define the path of the shear flow, q . This path will lie $a_o/2$ inside the centerline of the hoop as shown in Fig. 5. Knowing the path of the shear flow, the terms A_o (the

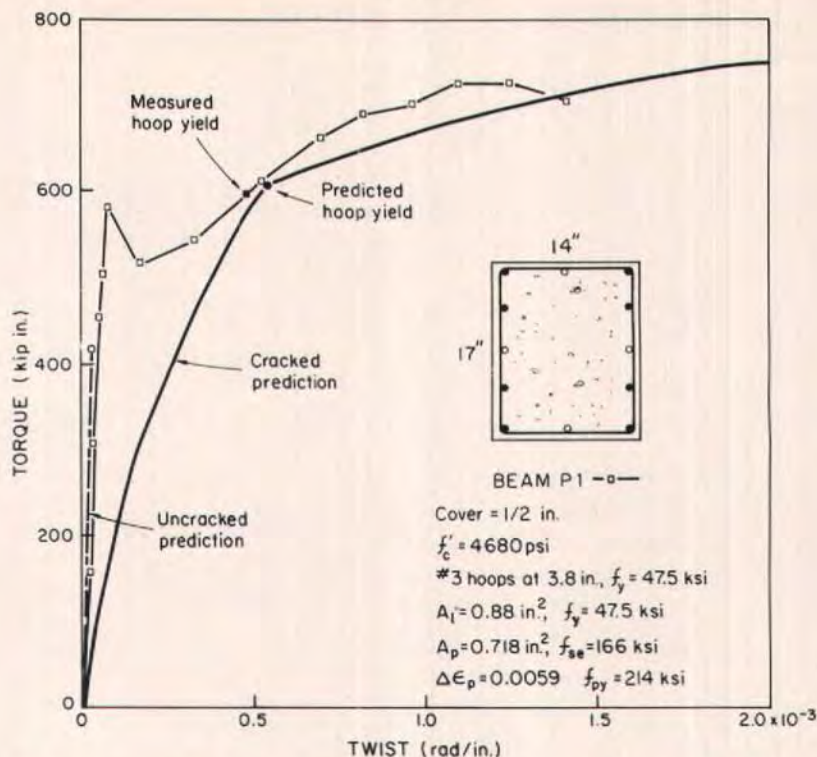


Fig. 7. Measured and predicted torque-twist response for a prestressed beam.

area enclosed by the shear flow) and p_o (the perimeter of the shear flow path) can be determined. Examination of Fig. 5 shows that A_o may be taken as:

$$A_o = A_{oh} - \frac{a_o}{2} p_h \quad (8)$$

where A_{oh} is the area enclosed by the centerline of the hoop and p_h is the hoop centerline perimeter. The perimeter of the shear flow path, p_o , can be taken as:

$$p_o = p_h - 4a_o \quad (9)$$

The area enclosed by the shear flow, A_o , for a variety of cross-sectional shapes is shown in Fig. 6.

As in flexure, the depth of compression

will be a function of the tensile forces in the reinforcement. It can be shown (see Appendix B) that:

$$a_o = \frac{\Delta N}{\alpha_1 f'_c p_o} + \frac{A_l f_t}{\alpha_1 f'_c s} \quad (10)$$

To illustrate the use of the compression field theory for torsion, the prediction of the torque-twist curve for the prestressed concrete beam shown in Fig. 7 will be described. The calculations would commence by choosing a value for the diagonal compressive strain at the surface of the concrete, ϵ_{ds} . Knowing ϵ_{ds} and the stress-strain curve of the concrete, the stress block factors α_1 and β_1 could be determined (Appendix A).

To determine the longitudinal and transverse strains in the beam which

correspond to the chosen value of ϵ_{ds} , it is convenient to rearrange the basic equations (see Appendix B) to give the following expressions:

$$\epsilon_t = \left[\frac{\alpha_1 \beta_1 f'_c A_{oh} s}{2 p_h A_t f_t} - 1 \right] \epsilon_{ds} \quad (11)$$

$$\epsilon_t = \left[\frac{\alpha_1 \beta_1 f'_c A_{oh} p_o}{2 p_h \Delta N} - 1 \right] \epsilon_{ds} \quad (12)$$

The strain in the prestressing steel is determined by adding the strain difference, $\Delta\epsilon_p$, to the longitudinal tensile strain, ϵ_t . When Eqs. (11) and (12) have been evaluated, then the tension in the hoop $A_t f_t$ and the tension in the longitudinal steel, ΔN , for the chosen value of ϵ_{ds} will be known. Eq. (10) can then be used to calculate the depth of compression, a_o , and hence the terms A_o and p_o .

Solving Eqs. (5) and (6) for the two unknowns, T and θ , gives:

$$T = 2A_o \sqrt{\frac{\Delta N A_t f_t}{p_o s}} \quad (13)$$

and

$$\tan \theta = \sqrt{\frac{A_t f_t p_o}{s \Delta N}} \quad (14)$$

Knowing θ , the twist of the beam for the chosen value of ϵ_{ds} can be determined by the geometrical relationship given in Fig. 5. Repeating the above calculations for different values of ϵ_{ds} enables the complete torque-twist response to be determined.

The torque-twist curve given by the solid line in Fig. 7 is based on the assumption that the concrete cannot resist any tensile stress. The dashed line represents the predicted precracking torque-twist response if tensile stresses in the concrete are taken into account. The torsion at which cracking occurs depends, of course, on the ten-

sile strength of the concrete and the level of prestress. An expression for this cracking torque will be given later in the paper.

If the observed¹⁸ experimental behavior of the beam shown in Fig. 7 is compared with the two theoretical predictions, it can be seen that prior to cracking the behavior closely follows the uncracked member prediction while after cracking the behavior tends towards the fully cracked member prediction.

To determine the ultimate torsional strength, it is not necessary to predict the complete torsional response. As in flexure, it can be assumed that the load which corresponds to a concrete strain of 0.003 is the maximum load the section can carry. When using this assumption to determine the torsional capacity, it is appropriate to use the ACI stress block factors.

Members in Shear

In applying the compression field theory to members in shear, it is again necessary to take some additional behavioral aspects into account. As in torsion, the unrestrained concrete cover may spall off at higher loads (see Fig. 8). Once again, it is assumed that after spalling the effective outer surface of the concrete will coincide with the centerline of the stirrups.

In calculating the diagonal compressive stress, f_d , it is assumed that the magnitude of the shear flow is constant over the effective shear depth d_v . Hence, the maximum value of the diagonal stress will occur at the location of the minimum effective web width, b_v , within the depth, d_v . In the truss analogy, d_v is the distance between the top and the bottom longitudinal chords. While d_v could be taken as the flexural lever arm jd , we will assume that d_v is the vertical distance between the centers of the longitudinal bars which are anchoring the ends of the stirrups.

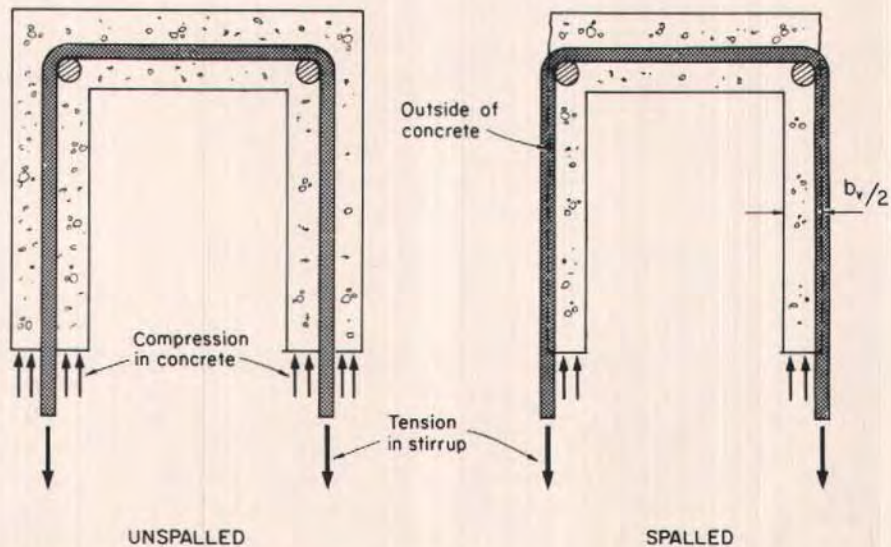


Fig. 8. Spalling of concrete cover due to shear.

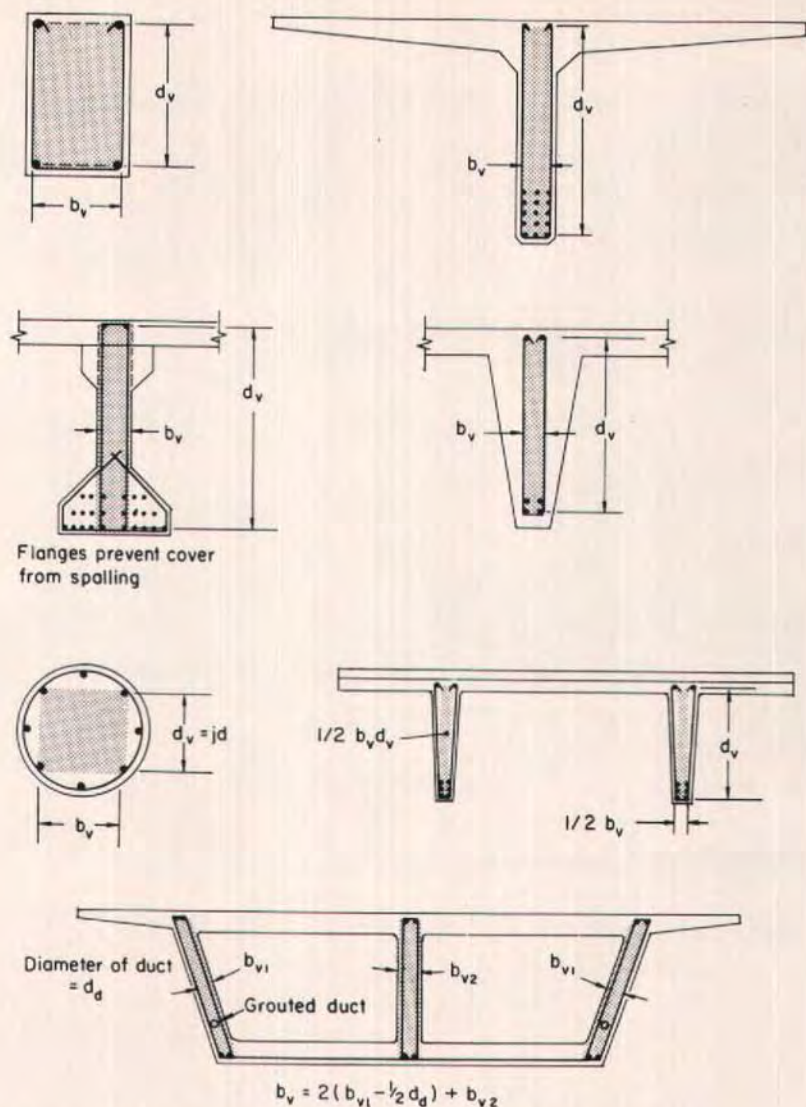


Fig. 9. Effective shear area of members with various cross sections.

The assumed effective areas resisting shear for a variety of cross-sectional shapes are shown in Fig. 9. The actual shear stress distributions for the cross sections shown will of course be non-uniform. However, the use of the effective shear areas shown in Fig. 9 will lead to conservative results. It has been observed^{19,20} that the presence of

large post-tensioning ducts in thin webs reduces the shear capacity. The suggested reduction in Fig. 9 comes from the CEB Code.⁸

The previously determined truss equations for shear plus the geometric relationship for θ can be used to determine the response of prestressed and non-prestressed concrete mem-

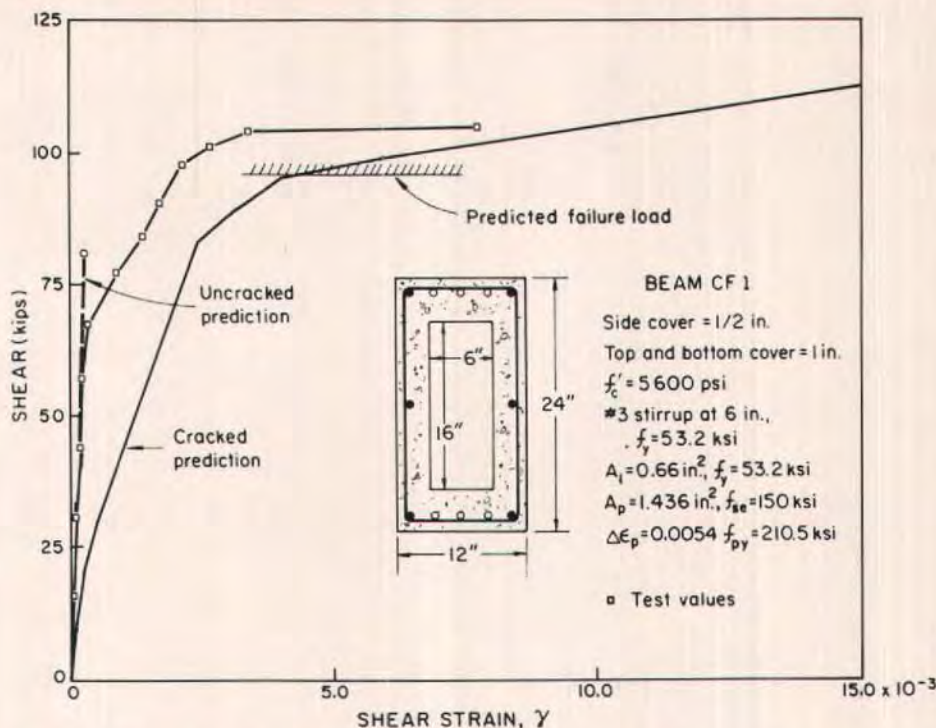


Fig. 10. Measured and predicted shear-shear strain response for a prestressed concrete beam.

bers in shear by the procedure already explained.

The resulting predicted shear force-shear strain response for a prestressed concrete box girder is shown in Fig. 10. The solid line represents the predicted response based on the assumption that concrete cannot resist any tensile stress. The predicted precracking response is represented by the dashed line. Once again, the observed²¹ experimental behavior follows the uncracked member prediction prior to cracking and tends towards the fully cracked member prediction after cracking.

In predicting the ultimate shear capacity of members, it has been found necessary to limit the maximum compressive stress, f_d . It must be appreciated that f_d is unlikely to reach

the cylinder crushing stress f'_c . Apart from the problems of the actual distribution of the principal compressive stresses (we have assumed a uniform stress distribution), this stress must be transmitted across cracked and severely deformed concrete.

Fig. 11 compares the failure conditions for the concrete in a test cylinder with the failure conditions for diagonally stressed concrete in a cracked beam loaded in shear. It has been proposed¹⁷ that the size of the stress circle that causes the concrete to fail is related to the size of the co-existing strain circle. As an indicator of the intensity of strain, the maximum shear strain, γ_m , (i.e., the diameter of the strain circle) is used. It has been suggested²² that the maximum value of f_d be taken as:

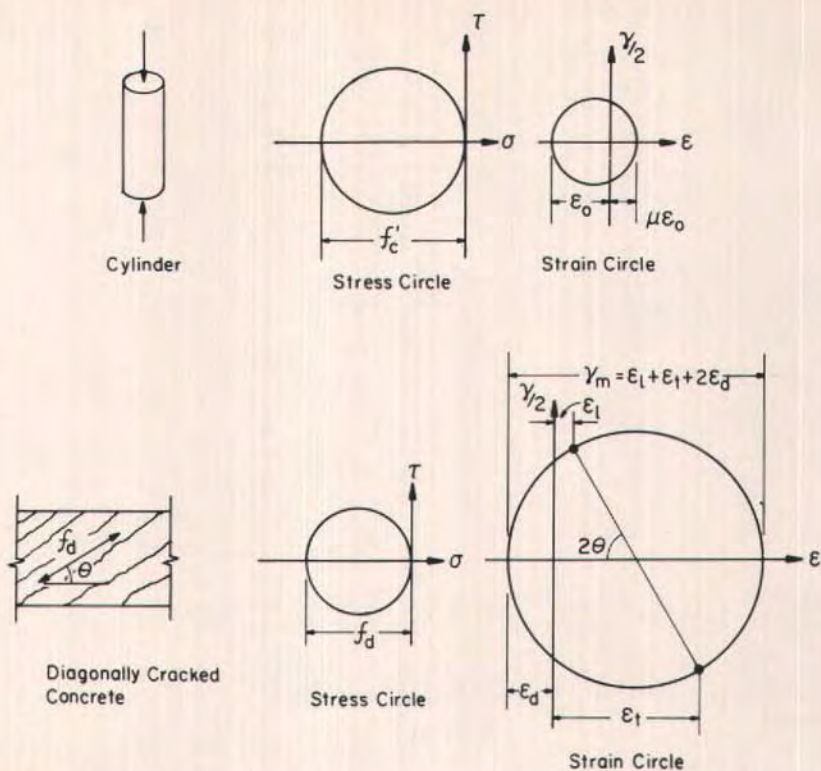


Fig. 11. Comparison of stress and strain conditions for a test cylinder and for diagonally cracked concrete.

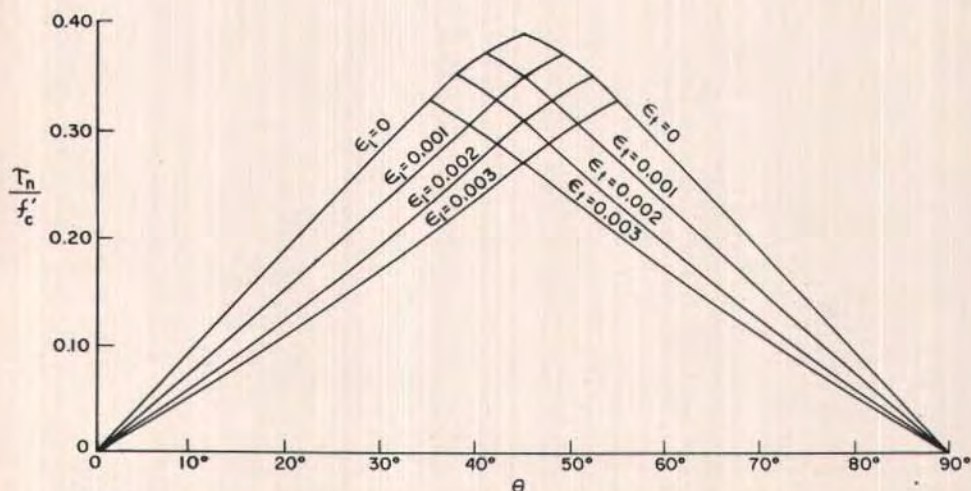


Fig. 12. Limits on angle of diagonal compression for torsion.

$$f_{du} = \frac{5.5 f'_c}{4 + \gamma_m / \epsilon_d} \quad (15)$$

where

$$\gamma_m = 2 \epsilon_d + \epsilon_t + \epsilon_l \quad (16)$$

and ϵ_d at failure is assumed to be 0.002, that is, the strain corresponding to the peak concrete stress. When the compressive stress f_d reaches the limiting value, f_{du} , failure is predicted. For the beam discussed above, the resulting predicted failure load is shown in Fig. 10. It can be seen that the prediction is conservative.

Design for Torsional Strength

In many practical situations the size and shape of the beam together with the amount of prestressed or non-prestressed longitudinal reinforcement will already have been chosen to satisfy other design considerations. The objectives of the torsion design then become:

- Check if the section size is adequate to resist the design torsion;
- Determine the area of transverse reinforcement required to resist the torsion; and
- Evaluate the additional longitudinal reinforcement required to resist the torsion.

If the section size is inadequate, the concrete will crush before the reinforcement yields. The transverse and longitudinal strains at the nominal torsional moment capacity T_n can be derived from Eqs. (5), (6), (11), and (12), as:

$$\epsilon_t = \left[\frac{0.85 \beta_1 f'_c A_o}{\tau_n A_{oh} \tan \theta} - 1 \right] 0.003 \quad (17)$$

$$\epsilon_l = \left[\frac{0.85 \beta_1 f'_c A_o}{\tau_n A_{oh}} \tan \theta - 1 \right] 0.003 \quad (18)$$

where

$$\tau_n = \frac{T_n \rho_h}{A_{oh}^2} \quad (19)$$

Before the above expressions can be used, it is necessary to evaluate the area enclosed by the shear flow, A_o . This area is a function of the depth of compression, a_o . By rearranging Eqs. (10), (5), (6), and (8), the following expression for a_o can be obtained:

$$a_o = \frac{A_{oh}}{\rho_h} \left[1 - \sqrt{1 - \frac{T_n \rho_h}{0.85 f'_c A_{oh}^2} \left(\tan \theta + \frac{1}{\tan \theta} \right)} \right] \quad (20)$$

It can be seen from the above equations that the strain conditions at ultimate depend on the angle θ .

The tasks of selecting an appropriate value of θ , and then calculating ϵ_t and ϵ_l are made considerably simpler by plotting Eqs. (17) and (18) in the form of the design chart shown in Fig. 12. In plotting the chart a value of 0.80 for β_1 was used.

As an example, let us assume the characteristics of the reinforcement are such that when $\epsilon_t = 0.002$ the transverse reinforcement will yield and when $\epsilon_l = 0.002$ the longitudinal reinforcement will "yield." For this case it can be seen from Fig. 12 that if τ_n / f'_c equals about 0.3, then both types of steel will yield only if θ is about 45 deg. On the other hand, if τ_n / f'_c equals about 0.1 both steels will yield for any value of θ between 16 and 74 deg.

The effect of choosing a lower value of θ is that, for a given torque, less hoop steel but more longitudinal steel will be required. Since hoop steel is typically more expensive than longitudinal steel, the design engineer may wish to use the lowest possible value of θ .

Having chosen an appropriate value of θ from Fig. 12, a_o can be determined from Eq. (20), the shear flow path parameters A_o and p_o can then be found from Eqs. (8) and (9), and finally the required areas of reinforcement can be calculated from Eqs. (5) and (6).

Designing for Combined Torsion and Flexure

The compression field theory described above is only strictly valid when the longitudinal strain ϵ_l is constant over the whole section which results in a constant value of θ . A more complex version of the compression field theory has been developed²³ which is capable of predicting the response of reinforced concrete beams under combined torsion and flexure. Fortunately, for under-reinforced beams a simple superposition procedure produces accurate results.

In this superposition procedure the transverse reinforcement is designed to resist the torsion by using the procedure already explained. The longitudinal reinforcement is then designed by the conventional plane sections theory to resist the applied moment plus the equivalent tension, ΔN , produced by the torsion.

To investigate the accuracy of the simple superposition method outlined above, it was used to predict the torsion-flexure interaction relationship for a series of uniformly prestressed, symmetrically reinforced concrete beams which had been tested at the University of Toronto.²⁴ The concrete cylinder strength varied somewhat between the five beams of the test series (individual values are shown in Fig. 13) so in the calculations the average value of 5.57 ksi (38 MPa) was used.

The predicted interaction curve was determined in the following manner:

For a chosen torque T the angle θ , which would result in the given amount of hoop reinforcement, was determined from Eqs. (6), (8), and (10). The axial tension ΔN , resulting from the torsion T with the calculated angle θ , was then found from Eq. (5). Finally, the moment M which the given section could resist in combination with the axial tension ΔN was determined from the conventional plane sections theory.

As can be seen from Fig. 13, the suggested procedure predicts accurately the observed strengths of the tested beams.

The predicted and observed cracking loads, which are shown in Fig. 13, will be discussed later in the paper.

The theory predicts that as the ratio of torsion to moment decreases, the angle of inclination of the diagonal compression θ will increase. Predicted values of θ range from 24 deg for pure torsion to 90 deg for pure flexure. While these predicted values of θ will not necessarily coincide with the average inclination of the cracks (they will coincide with the inclination of cracks which form just prior to failure), this inclination will provide an indication of the value of θ . If the crack patterns for the five tested beams, which are shown in Fig. 14, are studied, it can be seen that the crack inclinations are in reasonable agreement with the predicted values of θ .

In predicting the strengths of some of the beams shown in Fig. 13, it was necessary to apply the theory in regions where the longitudinal steel was not yielding. For example, in pure torsion $\theta = 24$ deg and $\tau_n/f'_c = 0.20$ which, as can be seen from Fig. 12, will correspond to $\epsilon_l = 0.6 \times 10^{-3}$.

When the capacity of such members is investigated by using the concept of an equivalent axial tension, allowance must be made for the fact that the longitudinal steel will not actually yield

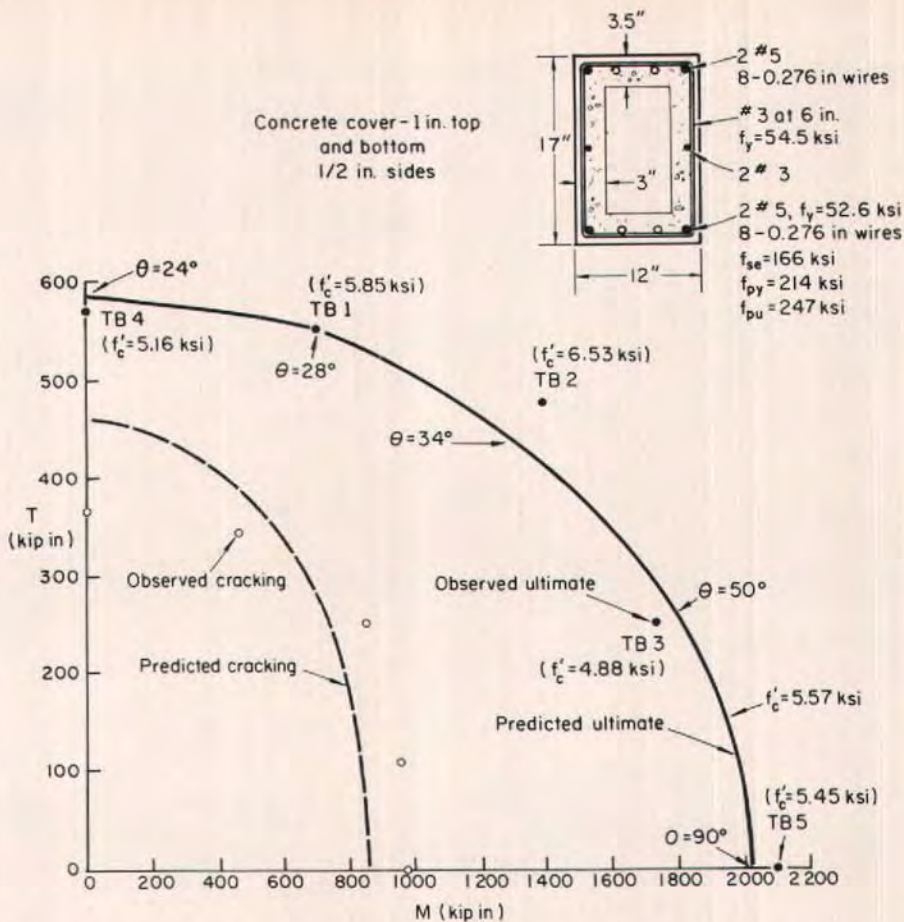


Fig. 13. Torsion-flexure interaction for a series of prestressed concrete beams.

under torsion while under direct tension the longitudinal steel would yield. To allow for this effect, the equivalent axial tension ΔN as given by Eq. (5) is increased by the ratio of the yield force of the longitudinal steel to the force in the steel at a strain of ϵ_{tj} .

Designing for Combined Shear and Flexure

Procedures analogous to those for torsion and flexure can be used to design beams subjected to shear and

flexure. As in torsion, it is necessary to check that the section is of an adequate size to resist the applied loads and then to determine the required amounts of transverse and longitudinal reinforcement.

To determine that the section size is adequate to resist the applied shear, it is necessary to check the transverse and longitudinal strains at ultimate. The following equations (see next page) which relate the strains at the nominal shear capacity, V_n , the angle θ , and the applied shear can be obtained by rearranging Eqs. (3), (15), (16), and (7).

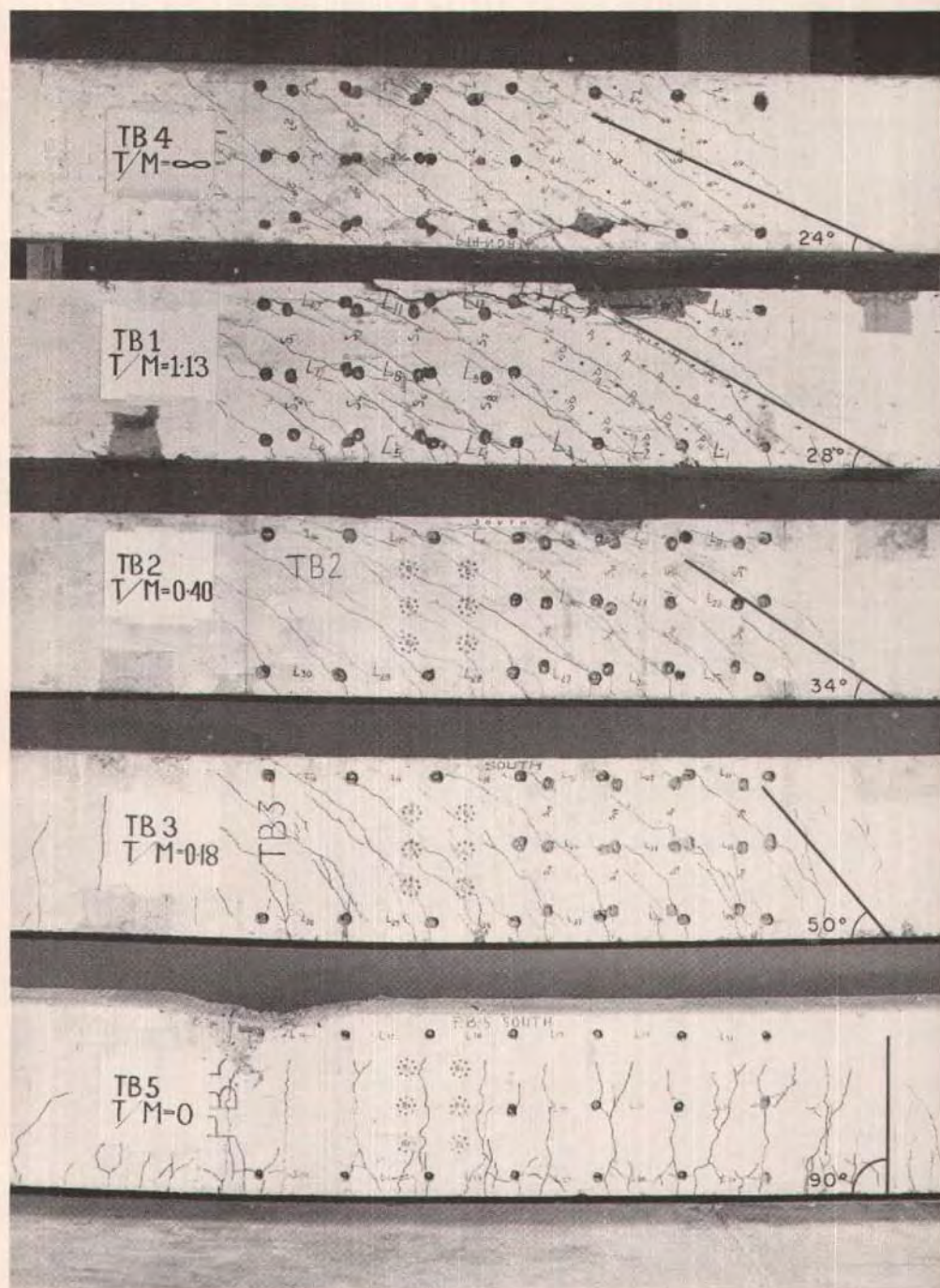


Fig. 14. Crack patterns for five prestressed concrete beams. (Specimens TB4, TB1, TB2, TB3, and TB5.)

$$\epsilon_t = \left[\frac{5.5 \sin\theta \cos\theta - 4 \frac{\tau_n}{f'_c}}{\frac{\tau_n}{f'_c} (1 + \tan^2\theta)} - 1 \right] 0.002 \quad (21)$$

$$\epsilon_t = \left[\frac{5.5 \sin\theta \cos\theta - 4 \frac{\tau_n}{f'_c}}{\frac{\tau_n}{f'_c} \left(1 + \frac{1}{\tan^2\theta} \right)} - 1 \right] 0.002 \quad (22)$$

where $\tau_n = V_n / (b_v d_v)$.

Once again, the tasks of selecting an appropriate value of θ and then determining if the section size is adequate are made considerably simpler if the above equations are plotted in the form of a design chart similar to Fig. 12. A modified form of this design chart will be presented in the following section.

Having chosen an appropriate value of θ , the required area of transverse reinforcement can be found from Eq. (3). The longitudinal reinforcement can then be designed by the plane sections theory to resist the applied moment plus the equivalent axial tension, ΔN , given by Eq. (1).

Designing for Combined Torsion Shear and Flexure

Although the compression field theory has not yet been fully extended to the case of beams loaded in combined torsion, shear, and flexure, a somewhat more approximate model called the variable angle space truss is available²⁵ and has been used as the basis of a computer-aided design procedure.²⁶ In this paper, an alternative simplified, conservative design procedure will be developed on the basis of the design charts for shear and torsion.

In Fig. 15 the design chart for torsion is compared with the design chart for shear. The design charts can be thought of as defining the limits of θ for a given level of stress and for given steel strains. It can be seen that the two design charts have somewhat similar shapes and that either chart could be reasonably represented by the following equation:

$$10 + \frac{\tau_n f'_c}{(0.42 - 50 \epsilon_t)} 35 < \theta < 80 - \frac{\tau_n f'_c}{(0.42 - 50 \epsilon_t)} 35 \quad (23)$$

where the angle θ is in degrees.

In applying the above design equation for the case of combined torsion and shear, we will take:

$$\tau_n = \frac{T_n p h}{A_{oh}^2} + \frac{V_n}{b_v d_v} \quad (24)$$

In designing a member for combined torsion and shear, the nominal shear stress τ_n would first be determined. Based on the yield strength of the reinforcement, appropriate values of ϵ_t and ϵ_c would then be calculated. Eq. (23) could then be used to calculate the range of possible values of θ . If the lower limit on θ is calculated to be higher than the upper limit on θ it

means that the section size is inadequate for the applied load.

After choosing an appropriate value of θ from within the allowable range (usually a value close to the lower limit of θ would be chosen), the amount of transverse reinforcement required to resist the shear could be determined from Eq. (3), while the amount of transverse steel needed to resist the torsion could be determined from Eqs. (6), (8), and (20). The required amount of transverse reinforcement is assumed to be the sum of the amount required for shear and the amount required for torsion.

If the shear alone were acting, then the longitudinal tension force required could be determined from Eq. (1). If on the other hand the torsion were acting alone, then the longitudinal tension force required could be determined by Eqs. (5), (8), and (20). If these two forces are simply added for the combined loading case, the result would be a conservative design. This is because on one face of the

member the torsion and the shear stresses counteract reducing the total longitudinal force required for equilibrium.

Comparisons with predictions from the variable angle space truss suggest that a simple conservative procedure for determining the required equivalent tension under combined loading is to take the square root of the sum of the squares of the individually calculated tensions. Thus, for combined loading the longitudinal reinforcement is designed to resist the applied moment and axial load plus an additional axial tension ΔN given by:

$$\Delta N_u = \frac{1}{\tan\theta} \sqrt{V_u^2 + \left(\frac{T_u p_o}{2A_o}\right)^2} \quad (25)$$

For members not subjected to an externally applied axial load, it may be more convenient to design for an equivalent additional moment rather than an additional axial load. The longitudinal reinforcement at a given section could thus be designed as

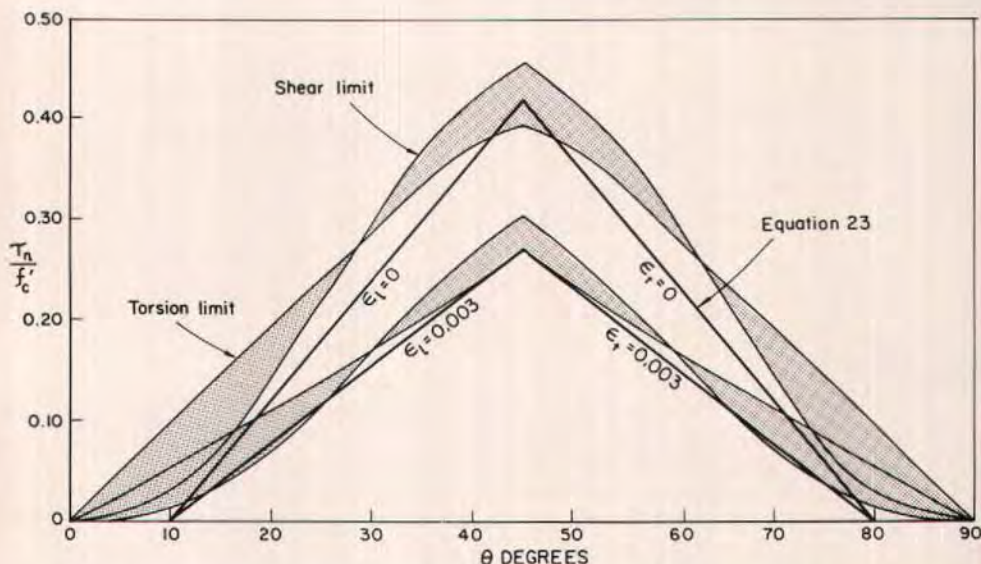


Fig. 15. Design limits on angle of diagonal compression for shear and torsion.

flexural tension reinforcement to resist a positive moment of $M_u + \frac{1}{2} d_v \Delta N_u$ and a negative moment of $\frac{1}{2} d_v \Delta N_u - M_u$. In regions of high positive moment, M_u would exceed $\frac{1}{2} d_v \Delta N_u$, indicating that top longitudinal reinforcement is not required.

Minimum Reinforcement Requirements

To ensure ductile behavior of flexural members, the ACI Code¹ requires that the amount of longitudinal reinforcement provided in prestressed concrete beams be large enough to ensure that the flexural capacity is at least 1.2 times the cracking moment. If the reinforcement is not capable of transmitting the cracking load, then the member may fail in a brittle manner when the first crack forms. To prevent such brittle failures for non-prestressed members, the ACI Code specifies a minimum percentage of flexural reinforcement. This minimum reinforcement is necessary unless the reinforcement provided is one-third greater than that required by analysis.

If the ACI Code philosophy for minimum flexural reinforcement is applied to members subjected to combined torsion, shear, and flexure, then either the reinforcement should be designed to transmit at least 1.2 times the cracking load or the reinforcement should be designed to transmit four-thirds of the factored design loads.

A simple, approximate procedure for calculating the cracking loads under combined loading can be developed from the following interaction equation:

$$\left(\frac{T_{ocr}}{T_{ocr}}\right)^2 + \left(\frac{V_{ocr}}{V_{ocr}}\right)^2 + \left(\frac{M_{ocr}}{M_{ocr}}\right)^2 = 1$$

where T_{ocr} , V_{ocr} , and M_{ocr} are the cracking loads under combined loading

while T_{ocr} is the pure torsional cracking load, V_{ocr} is the pure shear cracking load, and M_{ocr} is the pure flexural cracking load.

The pure torsional cracking load can be estimated²⁷ as:

$$T_{ocr} = \frac{A_c^2}{p_c} 4 \sqrt{f'_c} \sqrt{1 + \frac{f_{pe}}{4 \sqrt{f'_c}}} \quad (27)$$

where p_c is the outside perimeter of the concrete section, A_c is the area enclosed by p_c , and f_{pe} is the compressive stress due to prestress at the centroid of the section. In Eq. (27) f'_c is in psi units. If MPa units are used for f'_c the coefficients 4 should be replaced by 0.33.

The pure shear cracking load can be estimated¹ as:

$$V_{ocr} = b_w d 4 \sqrt{f'_c} \sqrt{1 + \frac{f_{pe}}{4 \sqrt{f'_c}}} \quad (28)$$

where b_w is the unspalled web width and d is the effective depth of the flexural steel. Once again, f'_c is in psi units. If MPa units are used, replace the 4's by 0.33.

The pure flexural cracking load can be estimated¹ as:

$$M_{ocr} = \frac{I}{y_t} (7.5 \sqrt{f'_c} + f_{pe}) \quad (29)$$

where I/y_t is the section modulus of the beam, f_{pe} is the compressive stress due to prestress at the extreme fiber of the section where tensile stress is caused by the applied moment, and f'_c is in psi. If MPa units are used, replace 7.5 by 0.6.

The above procedure was used to calculate the cracking loads for the prestressed concrete beams shown in Fig. 13. It can be seen that the observed cracking loads agree reasonably well with the predictions.

Control of Diagonal Cracking

Excessively wide cracks at service loads are undesirable. Obviously, cracking will not be a problem if the service load is lower than the cracking load. This will often be the case for concrete members which are prestressed. If the service load is greater than the cracking load, then it is necessary to provide an adequate amount of well detailed reinforcement to restrain the opening of the diagonal cracks.

In 1974 ASCE-ACI Shear Committee suggested²⁸ that "limiting the maximum stirrup strains at working loads to 0.001 . . . should prevent unsightly inclined cracks at working loads." In checking this suggested limitation, the compression field theory can be used to predict the strain in the transverse reinforcement at the specified service loads.

As can be seen in Fig. 16, the compression field theory, because it neglects concrete in tension, predicts that the transverse reinforcement will commence straining as soon as the load is applied. In reality, the transverse reinforcement will not begin to be strained until cracking occurs. After cracking there will be a transition between the uncracked condition and the fully cracked condition as the loads go above the cracking load.

Faced with an analogous problem in determining a stiffness which lies between the uncracked value and the fully cracked value, the ACI Code¹ uses an empirical transition formula based on the work by Branson.²⁹ This equation can be modified for our purposes to give the following expression:

$$\epsilon_{te} = \left[1 - \left(\frac{V_{cr}}{V} \right)^3 \right] \epsilon_t \quad (30)$$

in which ϵ_{te} is the expected transverse strain at the shear V and ϵ_t is the transverse strain predicted by the diagonal compression field theory.

The effect of this transition curve for a prestressed and a non-prestressed concrete beam is illustrated in Fig. 16. It can be seen that even if these two members have the same ultimate load, the prestressed member will have much smaller strains at service load levels. There are two reasons for this more desirable behavior of the prestressed concrete member; the cracking load is higher and the stiffness after cracking is larger.

In designing the section for ultimate loads, the choice of the angle θ determines the relative amounts of transverse and longitudinal steel. If a very low value of θ is chosen, only a very small amount of transverse steel will be supplied which may result in excessive stirrup strains at service loads. Thus, it is possible to think of the crack control limit as determining a lower limit on θ . Rearranging the basic equations and introducing some simplifying assumptions (see Appendix C) results in the following limit:

$$\tan\theta \geq \left(\frac{f_y V_{se}}{29 V_n} \right)^2 \left(1 - \frac{f_y f_{pc}}{29 f'_c} \right) \times \left[1 - \left(\frac{V_{cr}}{V_{se}} \right)^3 \right]^2 \quad (31)$$

where V_{se} is the shear at service load and f_y is the yield stress of the transverse steel in ksi units. If MPa units are used, the coefficients 29 should be replaced by 200.

Thus, in choosing the angle θ required to design the reinforcement at ultimate loads, the following two limit states must be considered:

1. To ensure that at ultimate the reinforcement yields prior to crushing of the concrete, θ must lie between the lower and upper limits given by Eq. (23).

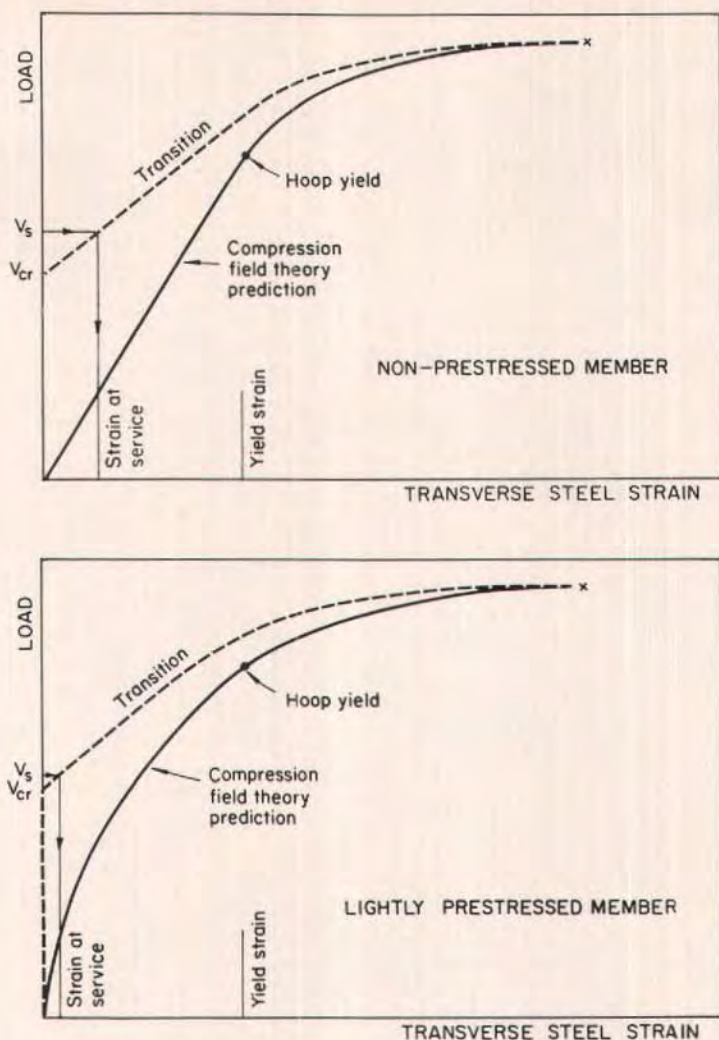


Fig. 16. Strain in transverse reinforcement at service load for a prestressed and non-prestressed member.

2. To ensure that at service loads the strain in the transverse reinforcement does not exceed 0.001, θ must be greater than the limit specified in Eq. (31).

These two limit states are compared in Fig. 17. In preparing this figure it was assumed that $f'_c = 5000$ psi (35 MPa), $V_{se}/V_n = 0.55$, $b_v d_v = 0.75 b_w d$, and that the cracking shears for the non-prestressed and prestressed

beams were $2\sqrt{f'_c} b_w d$ and $3.5\sqrt{f'_c} b_w d$, respectively (in MPa these values would be $0.17\sqrt{f'_c} b_w d$ and $0.29\sqrt{f'_c} b_w d$). It can be seen that for $f_y = 40$ ksi (300 MPa) the cracking limit on θ is not critical. It can also be seen that prestressing the beam makes the crack width limit on θ less restrictive.

In order to ensure control of diagonal cracking, it is necessary not only to

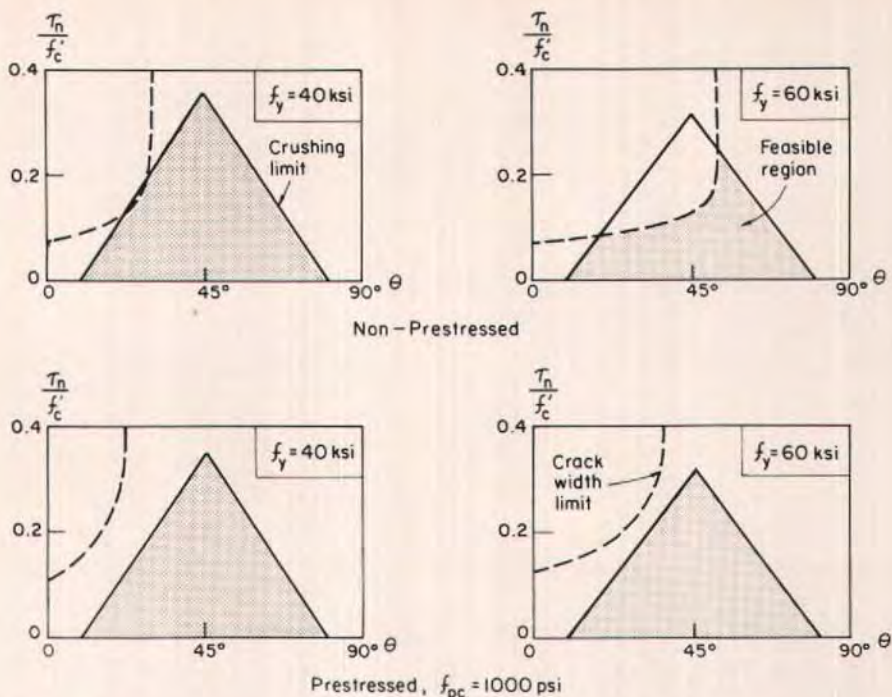


Fig. 17. Crushing limits and crack width limits for prestressed and non-prestressed concrete beams.

provide a sufficient amount of transverse reinforcement but also to limit the spacing of reinforcement so that an undesirable widening of the cracks between the reinforcement does not occur. It is suggested that provided both the longitudinal reinforcement and the transverse reinforcement have a spacing equal to or less than 12 in. (300 mm), this requirement will be satisfied. Since this requirement is concerned with controlling cracking at service loads, these spacing limits can be waived if the specified service loads are less than the cracking load.

Detailing of Reinforcement

In order that members subjected to shear and torsion perform satisfactorily, not only must an adequate

amount of reinforcement be provided but also it is essential that this reinforcement be correctly detailed. A rational model which enables the engineer to understand the required functions of the reinforcement will help the designer to avoid detailing mistakes, particularly in unusual situations not adequately covered by code rules.

A cracked beam in shear is idealized in Fig. 18(a) so that the functions of the reinforcement can be visualized more clearly. The primary function of the stirrups is to hold the beam together in the lateral direction. The distribution plates for the transverse steel shown in the idealized model enable the concentrated stirrup tensions to be distributed along the length of the beam balancing the outward thrusts of the diagonal concrete compressions.

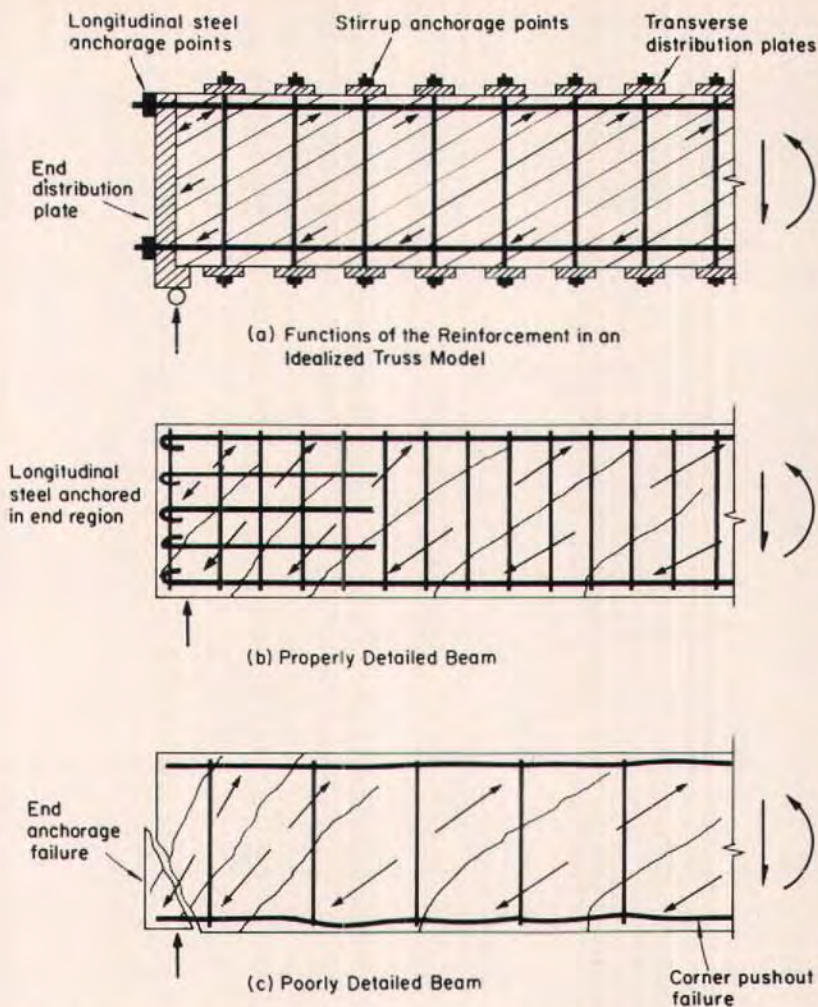


Fig. 18. Detailing considerations for a beam subjected to shear and/or torsion.

In a properly detailed beam [Fig. 18(b)], longitudinal bars at the anchor points of the stirrups perform the function of the distribution plates. In an improperly detailed beam [Fig. 18(c)] with excessive stirrup spacing, the longitudinal bars will not be capable of distributing the concentrated stirrup tensions and hence a premature failure may occur. For beams in torsion where the thrusts from two adjacent faces must be resisted (see Fig. 4), it is suggested that the diam-

eter of the longitudinal bars in the corners of the hoops should be at least $s \tan \theta / 16$.

The consequences of large stirrup spacing are further illustrated in Fig. 19. The compression field theory assumes a uniform distribution of diagonal compressive stresses over the beam as shown in Fig. 19(a). With widely spaced stirrups these diagonal stresses will not be uniform along the length of the beam but will become concentrated at the stirrup locations

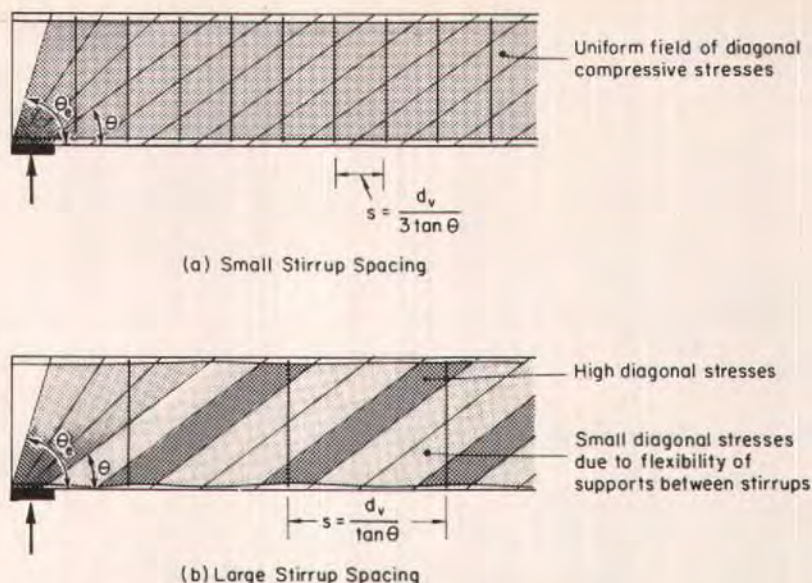


Fig. 19. Consequences of large stirrup spacing.

[see Fig. 19(b)]. These local concentrations of stress may result in premature diagonal crushing. To ensure a reasonably uniform distribution of stress, it is suggested that the stirrup spacing should not exceed $d_v/(3 \tan \theta)$. An analogous requirement for torsion would limit the spacing of the hoops to $p_h/(8 \tan \theta)$.

The three beams shown in Fig. 20 illustrate what happens when the spacing limits suggested above are exceeded. If Eqs. (3) and (23) are applied to these beams, which were among the many hundreds of beams tested at the University of Toronto by G. N. J. Kani,³⁰ a shear capacity of 14.9 kips (66.2 kN) and an angle θ of 18.7 deg would be predicted. The maximum spacing would thus be $d_v/(3 \tan 18.7) \approx d$. Beam 781, which had a maximum spacing of d , did not fail in shear. At a load corresponding to a shear of 13.4 kips (59.6 kN), the longitudinal steel in the central region of the beam yielded and the beam failed in flexure (see Fig. 20). Beam 782,

which had a maximum spacing of $2d$ and somewhat stronger flexural steel, failed in shear at 14.0 kips (62.2 kN) while Beam 783, with a maximum spacing of $3d$, failed in shear at 10.3 kips (45.8 kN).

As shown in the idealized model (Fig. 18), the transverse reinforcement must be properly anchored. The ACI Code¹ gives detailing requirements for the development of the transverse shear reinforcement. In applying these requirements, it should be kept in mind that at high shear stresses spalling of the unrestrained concrete cover may occur. In torsion, because of spalling of the concrete cover, considerable care must be taken to achieve proper end anchorage of the closed stirrups.³¹

In the idealized model the primary function of the longitudinal steel is to hold the beam together along its axis. The end distribution plate shown in the idealized model [Fig. 18(a)] enables the concentrated tensile forces in the longitudinal steel, which must

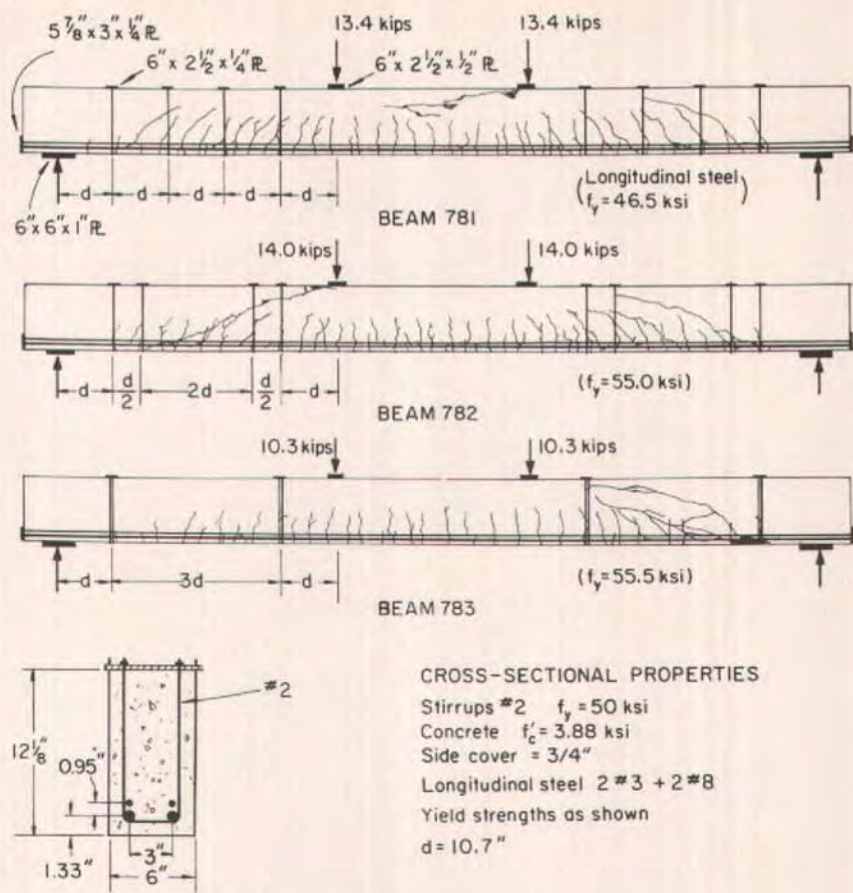


Fig. 20. Three beams with large stirrup spacings.

of course be anchored to the plate, to be distributed over the end of the beam. The tensile force in the steel balances the outward thrusts of the diagonal concrete stresses which are attempting to push off the end of the beam.

If the end of an actual beam frames into an adjacent member, then this member can act as the end distribution plate. Alternatively, end diaphragms or end blocks could act as end distribution plates. If none of these conditions are met, then care must be taken to provide proper end anchorage details. One such solution

involving well anchored and well distributed end longitudinal steel is shown in Fig. 18(b). Fig. 18(c) illustrates the consequences of not properly detailing the end anchorage region.

The beams shown in Fig. 19 do not have end blocks or well distributed longitudinal steel. For such cases the diagonal compressive stresses "fan-out" from the end bearing plate. The spread of the fan will be defined by the lowest angle of inclination, θ , and the highest angle of inclination, θ_e , of the diagonal compressive stresses.

If Eq. (23) is examined, it can be

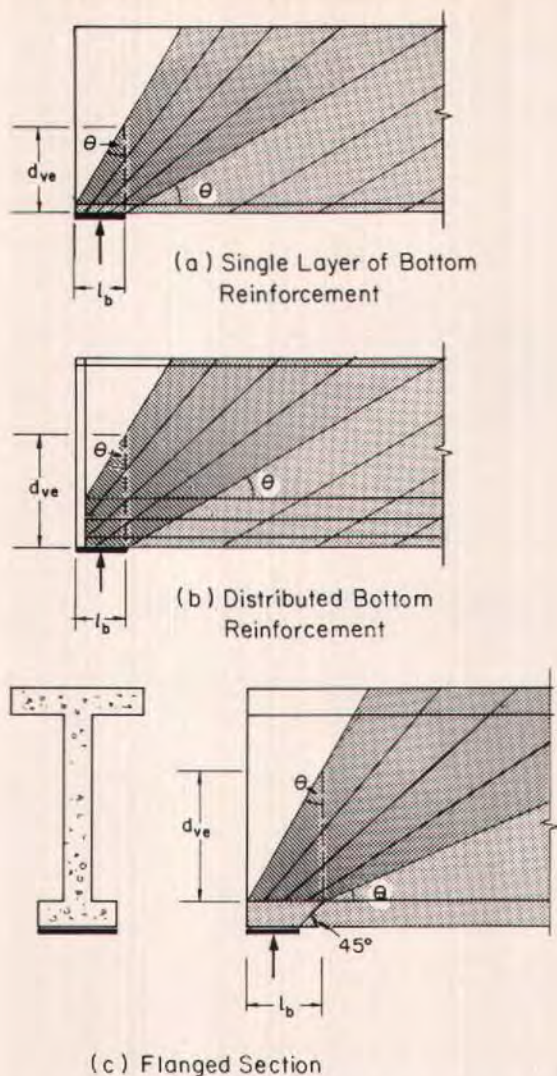


Fig. 21. Fanning of diagonal compression at ends of beams.

seen that for a given shear stress the sum of the lowest allowable angle and the highest allowable angle is approximately 90 deg. The two angles, θ and θ_e , defining the fan will usually be the lowest and the highest allowable angles, respectively, and hence it can be assumed that $\theta + \theta_e = 90$ deg.

The presence of the end fans (see Fig. 21) eliminates the need for ten-

sion in the top longitudinal steel near the end of the beam.³²

While the fan reduces the total longitudinal tensile force required in the steel at the end of the beam, it increases the compressive stresses in the concrete. In checking for crushing of the concrete in the end region, it can be assumed that the diagonal compressive stresses are distributed

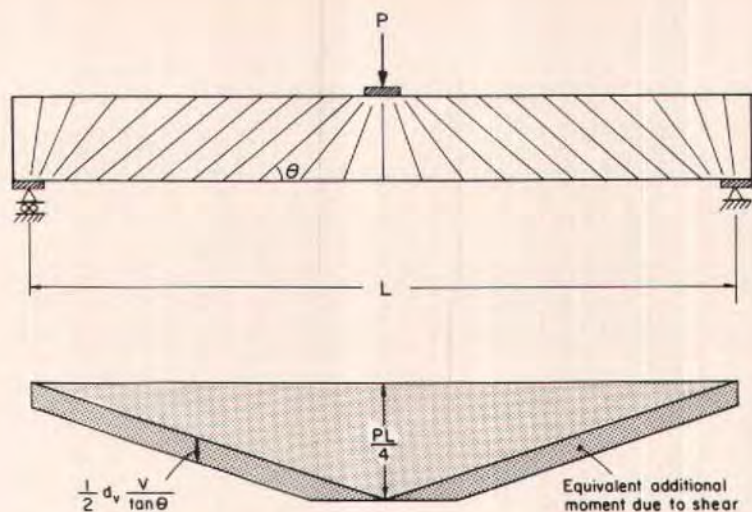


Fig. 22. Fanning of diagonal compression under concentrated load.

over the depth of the fan, d_{ve} , at the edge of the bearing plate (see Fig. 21).

The depth, d_{ve} , should be taken as $l_b \tan \theta_e = l_b / \tan \theta$, Fig. 21(a), unless a more detailed analysis indicates that well-anchored longitudinal steel spread over the depth of the beam, Fig. 21(b), enables d_{ve} to be increased. In checking web crushing for flanged members, the effective bearing length, l_b , can be increased since the critical section will no longer occur at the edge of the bearing plate [Fig. 21(c)].

If it is assumed that the tensile strains are negligible in the confined region near the bearing plate, then from Eq. (23) crushing will be avoided if:

$$\frac{V_n}{b_v d_{ve}} \leq \frac{(\theta - 10)}{35} 0.42 f'_c \quad (32)$$

One further effect caused by the fanning out of diagonal compressive stresses from a concentrated load is illustrated in Fig. 22. The tension in the longitudinal steel caused by shear decreases as the angle of inclination of

the diagonal compression increases. Directly under the load, where $\theta = 90$ deg, the shear will not cause any increase. As a consequence, the area of longitudinal steel in this region need not exceed the area required for the maximum flexure.^{8,32}

Distribution of Transverse Reinforcement

Design procedures to determine the required spacing of transverse reinforcement in regions of constant shear and/or torsion have already been explained. In regions of changing shears some additional factors need to be considered.

Fig. 23 compares three beams which have the same magnitudes of applied loading per unit length and hence the same shear force diagrams. The first beam has loading applied at the top face, the second has loading applied at the middle of the side face, while the third beam has the loading applied near the bottom face. For the free body diagrams shown, vertical

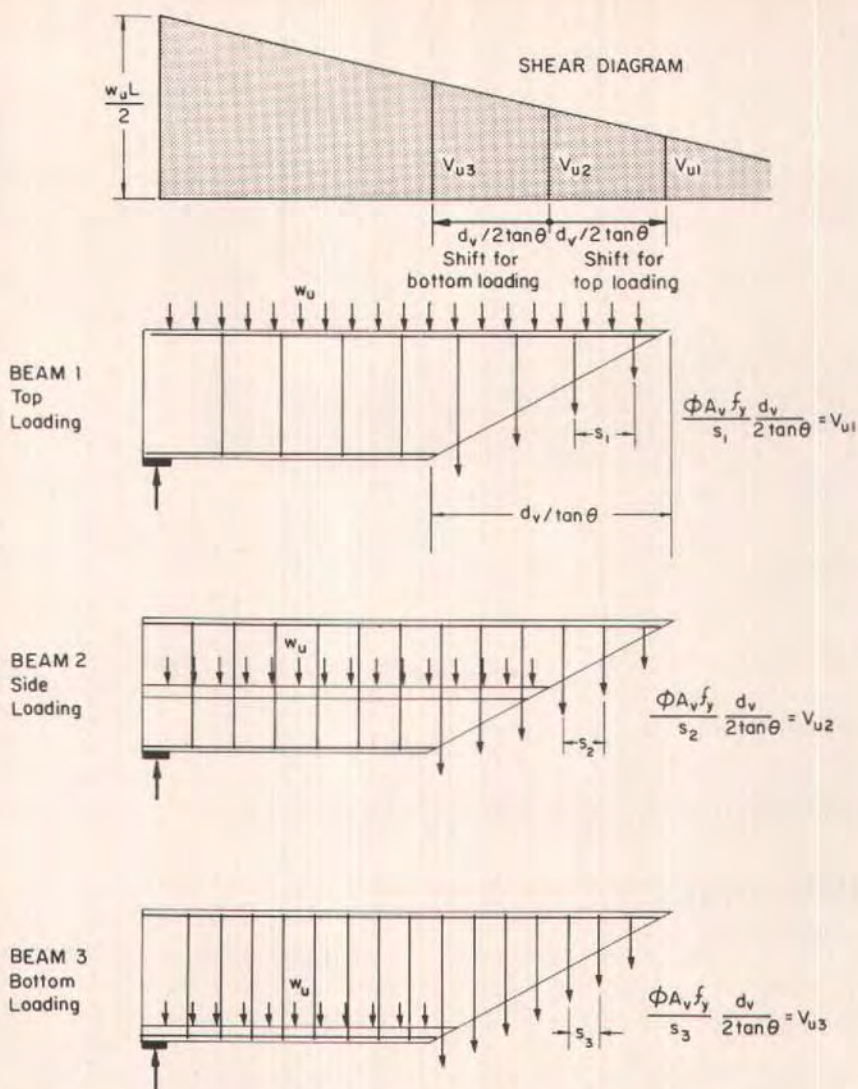


Fig. 23. Required shear strengths for top-loaded, side-loaded, and bottom-loaded beams.

equilibrium requires that the total tension in the stirrups crossed by the cut equals the end reaction minus the loading applied to the left of the cut. For Beam 2 this total force equals the shear at the midpoint of the cut. For Beam 1 this total force which the stirrups must resist is the reduced shear which occurs $d_v/(2 \tan \theta)$ to the right of

the midpoint of the cut. The increased shear which occurs, $d_v/(2 \tan \theta)$, to the left of the midpoint of the cut must be resisted by the stirrups in Beam 3.

Fig. 24 illustrates the way in which the distribution of transverse reinforcement is determined for a typical beam. From the given loading, the factored shear force diagram can be

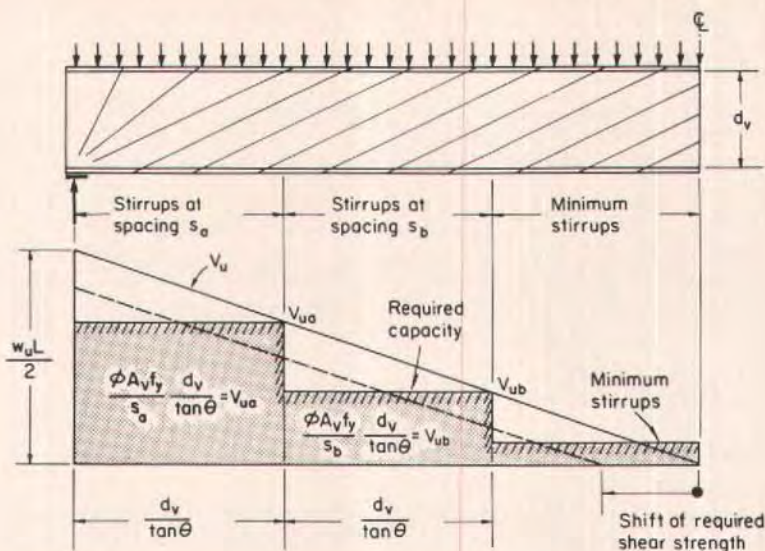


Fig. 24. Required transverse reinforcement for a uniformly loaded beam.

determined. For this top loaded beam, the shears for which the stirrups are to be designed are found by "shifting" the shear force diagram a distance of $d_v/(2 \tan \theta)$ towards the support as shown by the dashed line. Over the length $d_v/\tan \theta$ equilibrium will be satisfied if the stirrups are designed for the average shear force over this length.

The net effect of the shifting and the averaging of the shear forces for a

top loaded beam is that the transverse steel within a length, $d_v/\tan \theta$, is designed for the lowest factored shear, V_u , within this length. For a beam loaded near its bottom face, designing the transverse shear for this "lowest shear" would lead to insufficient reinforcement. For this case, it would be necessary to add additional transverse reinforcement capable of transmitting this load to the top face of the member.

* * *

PROPOSED DESIGN RECOMMENDATIONS FOR BEAMS IN SHEAR AND TORSION

The design procedures which have been discussed above have been summarized in the form of specific design recommendations. In formulating these recommendations some of the design equations were

generalized so that the recommendations could cover a wider range of practical problems. In particular, inclined stirrups, inclined prestressing tendons, and variable depth members have all been included.

1.0 — Notation

a_o	= equivalent depth of compression in torsion	f_{ps}	= stress in prestressing steel at nominal strength (see Section 18.7 of ACI 318-77)
A_c	= area enclosed by perimeter of cross section, p_c	f_y	= specified yield strength of non-prestressed reinforcement
A_g	= gross area of concrete cross section	I	= moment of inertia of gross concrete section resisting externally applied factored loads
A_o	= area enclosed by shear flow path	l_b	= effective length of bearing area taken as actual length of bearing except that for members with flanges where the bearing area is wider than the web l_b is taken as the actual length of bearing plus the vertical distance from the outer face of the flange to the junction of the web and the flange
A_{oh}	= area enclosed by hoop centerline	M_{cr}	= cracking moment under combined loading
A_t	= cross-sectional area of one leg of a closed stirrup	M_n	= nominal moment strength
A_v	= cross-sectional area of shear reinforcement within a distance s	M_{ocr}	= pure flexural cracking strength
b_v	= minimum effective web width within depth d_v (see Section 1.6.1)	M_u	= factored flexural moment
b_w	= minimum unspalled web width within depth d	N	= unfactored axial load
d	= distance from extreme compression fiber to centroid of longitudinal tension reinforcement	N_u	= factored axial load
d_v	= effective shear depth can be taken as flexural lever arm but need not be taken less than the vertical distance between centers of bars or prestressing tendons in corner of stirrups	ΔN_u	= equivalent factored axial load caused by shear and torsion
d_{ve}	= effective shear depth at end of beam	p_c	= outside perimeter of concrete cross section
E_s	= modulus of elasticity of steel	p_h	= perimeter of hoop centerline
f'_c	= specified compressive strength of concrete	p_o	= perimeter of shear flow path
f_{pc}	= compressive stress in concrete at centroid of cross section due to prestress (after allowance for all prestress losses)	s	= spacing of shear or torsion reinforcement in direction parallel to longitudinal axis
f_{pe}	= compressive stress in concrete due to effective prestress forces only (after allowance for all prestress losses) at extreme fiber of section where tensile stress is caused by externally applied loads	T_{cr}	= torsional cracking moment under combined loading
f_{po}	= stress in prestressing steel when strain in surrounding concrete is zero	T_n	= nominal torsional moment strength provided by circulatory shear flow
		T_{ocr}	= pure torsional cracking strength
		T_u	= factored torsional moment
		V_{cr}	= cracking shear under combined loading
		V_n	= nominal shear strength
		V_{ocr}	= pure shear cracking strength
		V_p	= vertical component of effective prestressing force or for variable depth members the sum of the vertical component of

the effective prestressing force and the vertical components of the flexural compression and tension

- V_{se} = service load shear
 V_u = factored shear force
 y_t = distance from centroidal axis of section to extreme fiber in tension
 α = angle between inclined stirrups and longitudinal axis of member
 β_v = factor accounting for non-yielding of the longitudinal steel under shear and/or torsion defined in Section 1.9.1
 β_1 = concrete stress block factor defined in Section 10.2.7 of ACI 318-77
 ϵ_t = tensile strain of longitudinal reinforcement due to shear and/or torsion
 ϵ_{tv} = yield strain of transverse reinforcement
 θ = angle of inclination to longitudinal axis of member (in degrees) of diagonal compressive stresses
 λ = factor to account for lightweight concrete defined in Section 1.4.1
 τ_n = nominal shear stress
 ϕ = strength reduction factor defined in Section 9.3 of ACI 318-77

1.1 — Scope

These recommendations are concerned with the design of prestressed and non-prestressed concrete beams subjected to shear or shear combined with torsion.

1.2 — General Principles and Requirements

1.2.1 — Beams shall be designed to have adequate strength, adequate ductility, and satisfactory performance at service load levels.

1.2.1.1 — To ensure adequate strength, the transverse reinforcement

shall be designed in accordance with Section 1.8 to resist the applied shear and torsion, the longitudinal reinforcement shall be designed in accordance with Section 1.9 to resist the applied flexure, shear, and torsion, while the section size shall be proportioned in accordance with Section 1.6 to avoid diagonal crushing of the concrete.

1.2.1.2 — To ensure adequate ductility, the members shall be designed to satisfy the minimum reinforcement requirements of Section 1.3 and the maximum reinforcement requirements specified in Section 1.6.

1.2.1.3 — To ensure adequate control of diagonal cracking at service load levels, members shall be designed to satisfy the requirements of Section 1.7.

1.2.2 — Reinforcement detailing requirements of Section 1.10 shall be satisfied.

1.3 — Minimum Reinforcement Requirements

1.3.1 — Amount of reinforcement in a member shall be chosen such that a reserve of strength exists after initial cracking.

1.3.2 — Requirement of Section 1.3.1 may be waived if the reinforcement is designed to resist factored loads one-third greater than those determined by analysis.

1.3.3 — The requirements of Section 1.3.1 will be satisfied if the amount of reinforcement at any section is such that the nominal sectional strengths, T_n , V_n , and M_n are at least equal to 1.2 times the cracking loads, T_{cr} , V_{cr} and M_{cr} , determined in accordance with Section 1.4.

1.3.4 — For members not subjected to moving loads, requirements of Section 1.3.3 need to be investigated only at locations of maximum moments.

1.4 — Cracking Loads

1.4.1 — In lieu of more exact analysis,

the cracking loads under combined flexure, shear, and torsion can be determined from the following interaction equation:

$$\left(\frac{M_{cr}}{M_{ocr}}\right)^2 + \left(\frac{V_{cr}}{V_{ocr}}\right)^2 + \left(\frac{T_{cr}}{T_{ocr}}\right)^2 = 1 \quad (1-1)$$

where

$$M_{ocr} = (I/y_t) (7.5\lambda \sqrt{f'_c} + f_{pe}) \quad (1-2)$$

(Stresses are in psi units; for MPa units replace 7.5 by 0.6.)

$$V_{ocr} = b_w d 4\lambda \sqrt{f'_c} \sqrt{1 + f_{pe}/(4\lambda \sqrt{f'_c})} + V_p \quad (1-3)$$

(Stresses are in psi units; for MPa units replace 4 by 0.33.)

$$T_{ocr} = (A_c^2/p_c) 4\lambda \sqrt{f'_c} \sqrt{1 + f_{pe}/(4\lambda \sqrt{f'_c})} \quad (1-4)$$

(Stresses are in psi units; for MPa units replace 4 by 0.33.)

where

- $\lambda = 1.0$ for normal weight concrete
- $\lambda = 0.75$ for "all-lightweight" concrete and
- $\lambda = 0.85$ for "sand-lightweight" concrete

1.4.2 — Eq. (1-4) can be used for hollow sections provided the least wall thickness is not less than $0.75 A_c/p_c$.

1.4.3 — In calculating the cracking loads, the ratio of torsion to shear, T_{cr}/V_{cr} , can be assumed equal to T_u/V_u .

1.4.4 — In calculating the cracking loads, the ratio of moment to shear, M_{cr}/V_{cr} , can be assumed equal to M_u/V_u . However, this ratio shall not be taken as less than d .

1.4.5 — Influence of axial loads on the magnitude of the cracking loads can be accounted for by increasing (for

compression) or decreasing (for tension) both f_{pc} and f_{pe} by N/A_g .

1.4.6 — For members not subjected to torsion or to axial load, cracking shear V_{cr} need not be taken as less than $2\lambda \sqrt{f'_c} b_w d$ (stresses in psi units; for MPa units replace 2 by 0.17).

1.5 — Consideration of Torsion

1.5.1 — If the magnitude of the factored torsional moment, T_u , as determined by an analysis using uncracked stiffness values exceeds $\phi(0.25 T_{ocr})$, then torsional reinforcement designed in accordance with Sections 1.8 and 1.9 shall be provided. Otherwise, torsional effects may be neglected.

1.5.2 — In a statically indeterminate structure where reduction of torsional moment in a member can occur due to redistribution of internal forces, the maximum factored torsional moment, T_u , may be reduced to $\phi(0.67 T_{ocr})$ provided that the corresponding adjustments to the moments and shears in adjoining members are made.

1.6 — Diagonal Crushing of the Concrete

1.6.1 — Nominal shear stress shall be computed as:

$$\tau_n = \frac{(V_n - V_p)}{b_v d_v} + \frac{T_n p_h}{A_o^2 h} \quad (1-5)$$

1.6.1.1 — In determining b_v , the unrestrained concrete cover down to the centerline of the outer transverse reinforcing bar shall be assumed to have spalled off; however, b_v need not be taken less than one-half of the minimum unspalled web width, b_w .

1.6.1.2 — In determining the minimum effective web width, b_v , the diameters of ungrouted ducts or one-half the diameters of grouted ducts shall be subtracted from the web width at the level of these ducts.

1.6.2 — Cross-sectional properties shall be chosen such that the trans-

verse reinforcement will yield prior to diagonal crushing of the concrete.

1.6.3 — Requirements of Section 1.6.2 may be considered satisfied if it is possible to choose an angle θ within the following limits:

$$10 + \frac{35(\tau_n/f'_c)}{(0.42 - 50\epsilon_t)} < \theta$$
$$< 80 - \frac{35(\tau_n/f'_c)}{(0.42 - 65\epsilon_{ty})} \quad (1-6)$$

where the value of ϵ_t can be chosen. However, the selected value of ϵ_t must also be used in satisfying the requirements of Section 1.9.1.

1.7 — Control of Diagonal Cracking

1.7.1 — Cross-sectional properties shall be chosen to ensure adequate control of diagonal cracking at service loads.

1.7.2 — Requirements of Section 1.7.1 may be considered satisfied if the cracking loads as determined by the procedures of Section 1.4 exceed the service loads.

1.7.3 — For uniformly loaded simply supported beams, crack control requirements need only be investigated for sections one-quarter of the span length from the supports.

1.7.4 — Requirements of Section 1.7.1 may be considered satisfied if the following three conditions are met:

- Calculations show that the strain in the transverse reinforcement at service loads does not exceed 0.001.
- Spacing of transverse reinforcement does not exceed 12 in. (300 mm).
- Spacing between longitudinal reinforcing bars at the cracked faces of the member does not exceed 12 in. (300 mm).

1.7.4.1 — Requirements of Section 1.7.4(a) may be considered satisfied if

in calculating the area of transverse reinforcement either the value of f_y is taken as not greater than 40 ksi (300 MPa), or the value of θ is such that:

$$\tan\theta \geq \left[\frac{f_y V_{se}}{29 V_n} \right]^2 \left[1 - \frac{f_y f_{pc}}{29 f'_c} \right]$$
$$\times \left[1 - \left(\frac{V_{cr}}{V_{se}} \right)^3 \right]^2 \quad (1-7)$$

where f_y is in ksi units (for MPa units replace 29 by 200).

1.7.4.2 — Bonded prestressing tendons may be considered equivalent to longitudinal reinforcing bars in satisfying the requirements of Section 1.7.4(c).

1.8 — Design of Transverse Reinforcement

1.8.1 — Transverse reinforcement provided for shear shall satisfy the detailing requirements of Section 1.10 and may consist of:

- Stirrups perpendicular to the axis of the member;
- Welded wire fabric with wires located perpendicular to the axis of the member;
- Stirrups making an angle of 45 deg or more with the longitudinal axis of the member.

1.8.2 — Transverse reinforcement provided for torsion shall satisfy the detailing requirements of Section 1.10 and may consist of:

- Closed stirrups perpendicular to the axis of the member;
- A closed cage of welded wire fabric with wires located perpendicular to the axis of the member;
- Spirals.

1.8.3 — Transverse reinforcement provided shall be at least equal to the sum of that required for shear and that required for torsion.

1.8.4 — In determining the required areas of transverse reinforcement, the

values of θ chosen must satisfy the limits specified in Eq. (1-6).

1.8.5 — Transverse reinforcement required for shear shall be determined from the requirement that $V_u \leq \phi V_n$.

1.8.6 — When shear reinforcement perpendicular to the axis of the member is used:

$$V_n = \frac{A_v f_y}{s} \frac{d_v}{\tan \theta} + V_p \quad (1-8)$$

1.8.7 — When inclined stirrups are used as shear reinforcement:

$$V_n = \frac{A_v f_y}{s} d_v \left(\frac{\sin \alpha}{\tan \theta} + \cos \alpha \right) + V_p \quad (1-9)$$

1.8.8 — Transverse reinforcement required for torsion shall be determined from the requirement that $T_u \leq \phi T_n$.

1.8.9 — Nominal torsional moment strength shall be computed by:

$$T_n = \frac{A_t f_y}{s} \frac{2A_o}{\tan \theta} \quad (1-10)$$

1.8.10 — The area enclosed by the torsional shear flow, A_o , may be computed as $A_{oh} - a_o p_h / 2$.

1.8.11 — The equivalent depth of compression in torsion, a_o , may be computed as:

$$a_o = \frac{A_{oh}}{p_h} \left[1 - \sqrt{1 - \frac{T_n p_h}{0.85 f'_c A_{oh}^2} \left(\tan \theta + \frac{1}{\tan \theta} \right)} \right] \quad (1-11)$$

1.8.12 — For hollow sections in torsion, the smallest thickness from the centerline of the stirrup to the inside face of the wall shall not be less than a_o / β_1 .

1.8.13 — For uniformly loaded beams

where V_u and T_u change linearly, the transverse reinforcement required within a length $d_v / \tan \theta$ may be determined by using the lowest values of V_u and T_u which occur within this length provided that θ is chosen to satisfy Eq. (1-6) by using in determining τ_n the highest values of V_u and T_u which occur within this length.

1.8.14 — For those beams for which the provisions of Section 1.8.13 do not apply, the transverse reinforcement required within a length $d_v / \tan \theta$ may be determined by using the average values of A_v / s and A_t / s calculated from Eqs. (1-8) or (1-9) and (1-10).

1.8.15 — In regions near supports which introduce direct compression into the member, the transverse reinforcement may be designed by using the values of V_u and T_u which occur at a distance $d_v / (2 \tan \theta)$ from the face of the support provided that θ is chosen to satisfy Eq. (1-6) by using in determining τ_n the values of V_u and T_u which occur at the face of the support.

1.8.16 — When downwards load is applied at the bottom face of a beam additional transverse reinforcement, capable of transmitting in tension the applied load to the opposite face of the beam, shall be provided.

1.8.17 — When a downwards load is applied to the side faces of a beam, then additional transverse reinforcement shall be provided. The amount of additional transverse reinforcement required may be assumed to vary linearly with the position of load on the side face going from zero when the load is at the top to the amount specified in Section 1.8.16 when the load is at the bottom.

1.9 — Design of Longitudinal Reinforcement

1.9.1 — The longitudinal reinforcement shall be designed by the plane sections theory described in Chapter 10 of ACI 318-77 to resist the factored moment, M_u , and the factored axial

load, N_u , together with an additional factored axial tension, ΔN_u , acting at mid-depth of the stirrups and given by:

$$\Delta N_u = \frac{\beta_v}{\tan\theta} \sqrt{(V_u - \phi V_p)^2 + \left(\frac{T_u p_o}{2A_o}\right)^2} \quad (1-12)$$

where β_v is a function of the value of ϵ_t used in satisfying Eq. (1-6) and θ is the value used in the design of the transverse reinforcement.

1.9.1.1 — For non-prestressed beams, β_v can be taken as $f_{yv}/(E_s \epsilon_t)$ but not less than one.

1.9.1.2 — For prestressed beams, β_v can be taken as $f_{pv}/(f_{po} + E_s \epsilon_t)$ but not less than one.

1.9.1.3 — Perimeter of shear flow path, p_o , may be computed as $p_h - 4a_o$.

1.9.2 — The ratios of longitudinal reinforcement shall be such that the requirements of Sections 10.3.3 and 18.8.1 of ACI 318-77 are satisfied.

1.9.3 — For members not subjected to axial load ($N_u = 0$), the requirements of Section 1.9.1 will be satisfied if the section is capable of resisting a factored moment equal to $M_u \pm \frac{1}{2} d_v \Delta N_u$.

1.9.4 — When an interior support or a concentrated load responsible for more than 50 percent of the shear at its location introduces direct compression into the flexural compression face of a member, then for those sections closer than $d_v/\tan\theta$ to the support or the load, the area of longitudinal reinforcement on the tension side need not exceed the area required to resist the M_u and N_u which exist at the nearest section where maximum moments occur. However, if torsion is present, then the sections must be capable of resisting an additional tension force of $T_u p_o/(2A_o \tan\theta)$ at the mid-depth of the stirrups.

1.9.5 — At the loaded free ends of cantilevers or at the ends of simply supported beams where the loads or reactions introduce compression into the end regions, the longitudinal reinforcement in these regions shall be designed according to the following provisions.

1.9.5.1 — The longitudinal reinforcement near the flexural tension face shall be anchored such that at the inner edge of the bearing area a factored moment of $M_u + \frac{1}{2} d_v \Delta N_u$ can be resisted.

1.9.5.2 — For those sections closer than $d_v/\tan\theta$ to the inner edge of the bearing area, the longitudinal reinforcement near the flexural compression face shall be designed such that at each section a factored moment of:

$$\frac{d_v}{2} \frac{\beta_v}{\tan\theta} \frac{T_u p_o}{2A_o} - M_u$$

causing tension on the flexural compression face can be resisted.

1.9.5.3 — Axial tensile loads, N_u , applied to ends of cantilevers or beams shall be resisted by appropriately anchored additional longitudinal reinforcement.

1.9.5.4 — The cross-sectional properties and bearing area dimensions shall be such that at the inner edge of the bearing area:

$$\frac{V_n - V_p}{b_v d_{ve}} + \frac{T_n p_h}{A_o^2} \leq 0.012 (\theta - 10) f'_c \quad (1-13)$$

where θ is the angle used in determining the amount of transverse reinforcement. If θ is greater than 45 deg, the term $(\theta - 10)$ shall be replaced by $(80 - \theta)$. The term d_{ve} shall be taken as $l_b/\tan\theta$ unless a more detailed analysis indicates that well anchored longitudinal reinforcement spread over the depth of the beam enables d_{ve} to be increased.

1.10 — Reinforcement Details

1.10.1 — Transverse shear and torsion reinforcement shall be anchored according to Section 12.14 of ACI 318-77 except that in regions of unconfined concrete cover, torsion reinforcement shall be anchored by means of 135-deg standard hooks.

1.10.2 — A longitudinal reinforcing bar or prestressing tendon shall be provided at each interior corner of transverse reinforcement. The nominal diameter of this bar or tendon shall be at least equal to the diameter of the stirrup but shall not be less than 1/2 in. (12 mm).

1.10.2.1 — Nominal diameter of the bar or tendon provided in each corner of closed transverse reinforcement re-

quired for torsion shall not be less than $s \tan\theta/16$.

1.10.3 — Where shear reinforcement is required, its spacing shall not exceed $d_v/(3 \tan\theta)$.

1.10.4 — Where torsion reinforcement is required, its spacing shall not exceed $p_h/(8 \tan\theta)$.

1.10.5 — Except at supports of simple spans and at free ends of cantilevers, the longitudinal reinforcement shall extend beyond the point at which it is no longer required to resist any stress for a distance equal to twelve times the bar diameter. The additional requirement of Section 12.11.3 of ACI 318-77 to extend this reinforcement for a distance equal to the effective depth of the member may be waived.

* * *

Comparisons With Other Design Procedures

Fig. 25 compares the predictions of the compression field theory (CFT), the ACI Code, and the CEB Code for four different series of beams. The compared predictions relate the amount of transverse reinforcement (expressed by the non-dimensional ratio, $A_n f_y / (b_w s f'_c)$) to the resulting shear strength (expressed by the non-dimensional ratio, τ_n / f'_c).

In preparing this figure, it was assumed that $f'_c = 5000$ psi (35 MPa), $V_w/V_n = 0.55$, $b_v = 0.8 b_w$, $d_v = 0.9 d$, and that the cracking shears for the non-prestressed and prestressed beams were $2\sqrt{f'_c} b_w d$ and $3.5\sqrt{f'_c} b_w d$, respectively ($0.17\sqrt{f'_c} b_w d$ and $0.29\sqrt{f'_c} b_w d$ for MPa units).

To facilitate comparisons between the ACI and the CEB approaches, the material strength reduction factors

used in the CEB Code were taken as unity. Further, the crack control requirements of the CEB Code which are expressed in terms of the stirrup spacing were not applied.

It can be seen that for lightly reinforced beams, the three methods give quite similar results. However, for members subjected to very high shear stresses there is a considerable divergence between the predictions of the three methods. It is worth noting that the ACI upper limit was primarily intended to control cracking at service loads. For the non-prestressed beams with the higher strength reinforcement, where the compression field theory predicts that cracking at service loads will control, this ACI upper limit looks appropriate.

Further comparisons between the reinforcement required by the compression field theory and that required by other design methods are given in the three design examples at the end

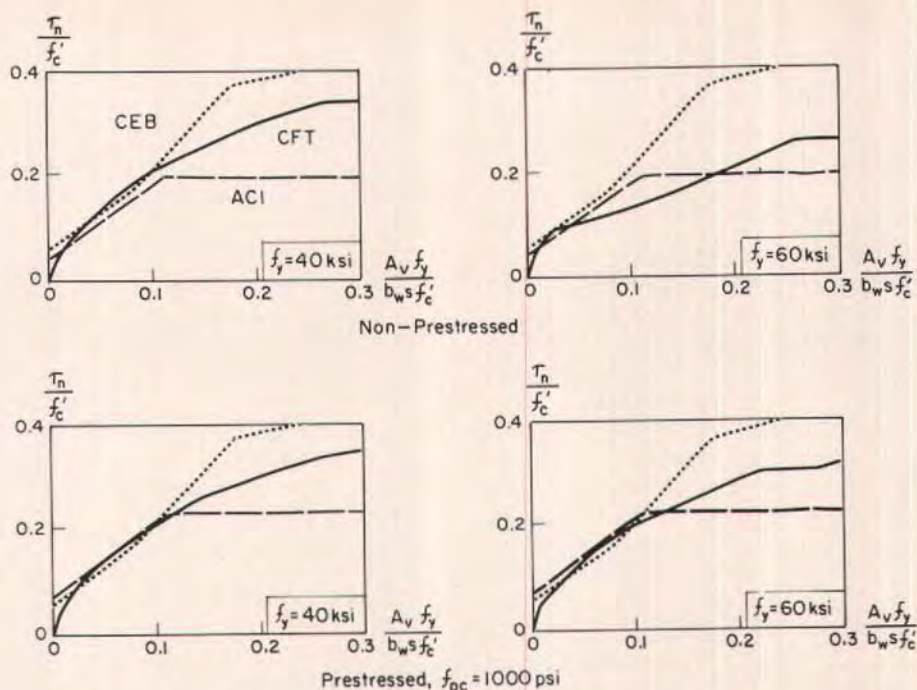


Fig. 25. Relationships between amounts of shear reinforcement and shear strength predicted by compression field theory, ACI Code, and CEB Code.

of this paper. From these examples it can be seen that the amount of reinforcement required by the compression field theory is in general comparable to the amounts required by existing empirical design procedures.

Concluding Remarks

This paper has presented design recommendations, based on the compression field theory, which enable prestressed and non-prestressed concrete beams to be designed for shear and for torsion.

The compression field theory, which is a development of the traditional truss model for shear and torsion, considers in addition to the truss equilibrium conditions, geometric compatibility conditions and material

stress-strain relationships. Like the plane sections theory for flexure and axial load, the compression field theory is capable of predicting not only the failure load but also the complete load-deformation response.

The design recommendations developed from the compression field theory have the following features:

1. The expressions that relate the required areas of reinforcement to the design loads are the classical truss equilibrium equations.

2. Rather than setting the angle of diagonal compression, θ , to 45 deg as in traditional North American practice or choosing it within the empirical range of 31 to 59 deg as in recent European practice, the range of allowable angles is calculated based on considerations of concrete crushing and control of diagonal cracking. For

highly stressed members, θ will be restricted to a narrow range of values close to 45 deg while for lightly loaded members the range of allowable angles will be very wide (in the limit $10 \text{ deg} \leq \theta \leq 80 \text{ deg}$).

3. In determining shear and torsion capacities, the tensile strength of the concrete is ignored. Because θ can vary within such wide limits, it is not necessary to introduce an empirical correction term (a "concrete contribution") to account for the strength of lightly reinforced members.

4. Prestressed concrete beams, non-prestressed concrete beams, and partially prestressed concrete beams having a wide variety of cross-sectional shapes can all be designed using the same basic expressions. The beneficial effects of prestressing are accounted for in the design by allowing lower values of θ to be used which will result in less transverse reinforcement.

The compression field theory assumes that diagonal compressive stresses can be transmitted through cracked concrete. Transmitting diagonal compressive stresses across a crack which is inclined to these stresses will require shear stresses to be transmitted from one face of the crack to the other. These interface shear stresses are transmitted by aggregate interlock⁴ and by the dowel action of the reinforcement crossing the crack.

If an excessively wide crack forms

across the whole member depth, the mechanisms of "interface shear transfer"²⁸ will deteriorate and may cause a premature failure. For this reason the design recommendations should not be applied to sections subjected to reversed cyclic loading of such an intensity that excursions into the yield range of the flexural reinforcement are expected.

In certain situations the present recommendations may lead to overly conservative designs. Further research will probably result in refinements of the design recommendations in the following areas:

- Local effects caused by concentrated forces;
- Minimum reinforcement requirements;
- Effectiveness of large unrestrained concrete covers;
- Influence of variable web width;
- Effects of indirect loading; and
- Torsional resistance provided by restrained warping stresses.

The authors believe that the recommendations presented in this paper are more rational and more general than the shear and torsion provisions of current North American codes. Hopefully, the design method presented will enable engineers to develop a better understanding of the behavior of beams in shear and torsion, an understanding which should result in more efficient and economical designs.

* * *

DESIGN EXAMPLES

To illustrate the proposed design method, three numerical examples are presented: (1) single tee, (2) spandrel beam, and (3) bridge girder.

DESIGN EXAMPLE 1—PCI Standard Single Tee

The single tee floor beam shown in Fig. 26 has been constructed with sand-lightweight concrete weighing 120 pcf and topped with 2½ in. of normal weight concrete. The beam supports a live load of 75 psf and spans 70 ft.

1. Determine factored loads

Dead load:

$$\begin{aligned} \frac{570}{144} \times 120 + \frac{2.5}{12} \times 150 \times 8 \\ = 475 + 250 = 0.725 \text{ kips/ft} \end{aligned}$$

Live load:

$$8 \times 75 = 0.600 \text{ kips/ft}$$

Therefore, factored uniform load per unit length:

$$\begin{aligned} w_u = 1.4 \times 0.725 + 1.7 \times 0.600 \\ = 2.035 \text{ kips/ft} \end{aligned}$$

2. Check minimum reinforcement requirements (Section 1.3)

Since this member is not subjected to moving loads, it is only necessary to check Section 1.3.3 at midspan. Assume the effective prestress equals 150 ksi. Therefore, prestressing force:

$$P = 13 \times 0.153 \times 150 = 298 \text{ kips}$$

Hence, bottom fiber stress due to prestress:

$$\begin{aligned} f_{pe} &= \frac{P}{A} + \frac{Pe}{Z_b} \\ &= \frac{298}{570} + \frac{298 \times 23.01}{2650} \\ &= 3.110 \text{ ksi} \end{aligned}$$

Note that untopped section properties have been used in determining f_{pe} . The dead load of the topping plus the dead load of the precast beam will cause a moment at midspan of

$$\frac{0.725 \times 70^2}{8} \times 12 = 5330 \text{ kip-in.}$$

which will reduce the bottom fiber

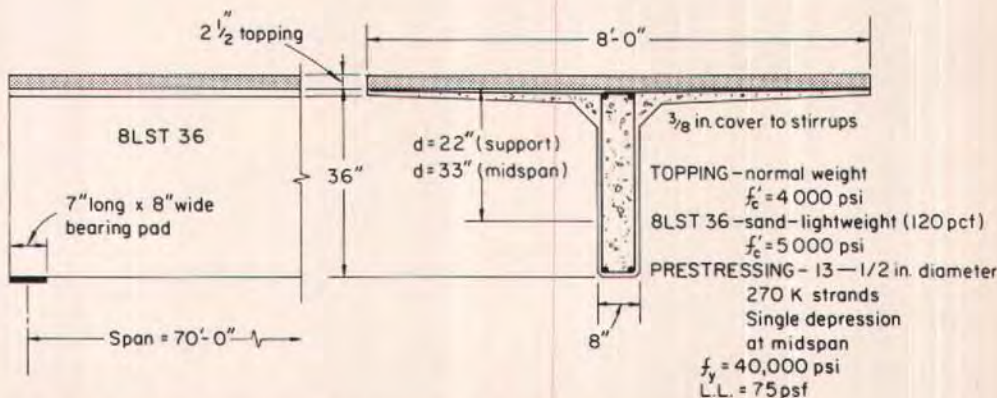


Fig. 26. Design Example 1—Design of a lightweight single-tee member.

precompression by $5330/2650 = 2.011$ ksi.

To crack the beam the bottom fiber must reach a tensile stress of

$$7.5\lambda \sqrt{f'_c} = 7.5 \times 0.85 \times \sqrt{5000} \\ = 0.451 \text{ ksi}$$

Hence, additional moment to crack beam:

$$3142 (3.110 - 2.011 + 0.451) = 4870 \text{ kip-in.}$$

(Note that 3142 in.³ is Z_b of composite section.)

Thus, cracking moment under combined loading:

$$M_{cr} = 5330 + 4870 = 10,200 \text{ kip-in.}$$

Factored flexural moment:

$$M_u = \frac{w_u L^2}{8} = \frac{2.035 \times 70^2}{8} \times 12 \\ = 14,960 \text{ kip-in.}$$

Hence, to provide the flexural strength required, the nominal moment strength:

$$M_n \geq M_u / \phi = 14,960 / 0.9 \\ = 16,620 \text{ kip-in.}$$

Therefore, the nominal to cracking moment ratio:

$$M_n / M_{cr} = 16,620 / 10,200 \\ = 1.63 > 1.20$$

Hence, minimum reinforcement requirements will be satisfied.

3. Check cross-sectional size and choose angle θ (Section 1.6)

Calculate the nominal shear stress at face of support. For simplicity, neglect the vertical component of the prestressing force, V_p , at this stage.

Factored shear force:

$$V_u = 2.035 \times 35 = 71.2 \text{ kips}$$

Nominal shear strength:

$$V_n = 71.2 / 0.85 = 83.8 \text{ kips}$$

Minimum effective web width:

$$b_e = 8 - 2 \times \frac{3}{8} - \frac{3}{8} = 6.9 \text{ in.}$$

(unrestrained cover neglected)

Effective shear depth:

$$d_e = 36 - 2 - 1 = 33 \text{ in.}$$

From Eq. (1-5), nominal shear stress:

$$\tau_n = \frac{V_n}{b_e d_e} = \frac{83.8}{6.9 \times 33} = 0.368 \text{ ksi}$$

Therefore, $\tau_n / f'_c = 0.368 / 5 = 0.074$.

For $f_y = 40$ ksi, yield strain of transverse reinforcement:

$$\epsilon_{ty} = 40 / 29,000 = 0.00138$$

Use $\epsilon_t = \epsilon_{ty}$.

To avoid crushing of the concrete, the angle of inclination, θ , must satisfy the limits given by Eq. (1-6). Thus:

$$10 + \frac{35 (\tau_n / f'_c)}{0.42 - 50 \epsilon_t} < \theta \\ < 80 - \frac{35 (\tau_n / f'_c)}{0.42 - 65 \epsilon_{ty}}$$

$$10 + \frac{35 \times 0.074}{0.42 - 50 \times 0.00138} < \theta \\ < 80 - \frac{35 \times 0.074}{0.42 - 65 \times 0.00138}$$

$$17.4 \text{ deg} < \theta < 72.2 \text{ deg}$$

Therefore, the section size is adequate. Choose $\theta = 20$ deg.

Although different values of θ could be used in designing different regions along the length of the beam, it will typically be more convenient to use the same value of θ over the length of the beam.

While the smallest allowable value of θ (in this case 17.4 deg) will result in the smallest amount of transverse steel, it may result in an excessively large amount of longitudinal steel. Hence, it is usually prudent to choose a value of θ somewhat larger than the smallest allowable angle.

4. Check diagonal crack control requirements (Section 1.7)

Compare cracking loads (Section 1.4) and service loads at $L/4$ from support (Section 1.7.3).

Calculate the service load shear at $L/4$ from support:

$$V_{se} = 0.5 \times 35 (0.725 + 0.600) = 23.2 \text{ kips}$$

Calculate the shear required to cause diagonal cracking. For simplicity use Section 1.4.6:

$$\begin{aligned} V_{cr} &= 2\lambda \sqrt{f'_c} b_w d \\ &= 2 \times 0.85 \sqrt{5000} \times 8 \times \\ &\quad \left(\frac{22 + 33}{2} + 2.5 \right) \\ &= 28.9 \text{ kips} \end{aligned}$$

Since $V_{cr} > V_{se}$, the crack control requirements are satisfied.

5. Design of transverse reinforcement (Section 1.8)

For a given shear the transverse reinforcement required can be determined by rearranging Eq. (1-8) to give:

$$\frac{A_v f_y}{s} \frac{d_v}{\tan \theta} \geq \frac{V_u}{\phi} - V_p$$

If Grade 40 #3 U stirrups are used, then:

$$\frac{A_v f_y}{s} \frac{d_v}{\tan \theta} = \frac{0.22 \times 40 \times 33}{s \tan 20} = \frac{798}{s}$$

For this uniformly loaded beam the transverse reinforcement within each length of $d_v/\tan \theta$ is designed for the lowest value of shear within this length (Section 1.8.13).

$$\frac{d_v}{\tan \theta} = \frac{33}{\tan 20} = 90.7 \text{ in. (or 7.6 ft)}$$

This results in the "stepped" requirement for transverse steel shown in Fig. 27. The stirrup spacings chosen to satisfy the requirements are also shown in Fig. 27.

Note that maximum spacing of shear reinforcement (Section 1.10.3) is

$$d_v/(3 \tan \theta) = 33/(3 \times \tan 20) = 30.2 \text{ in.}$$

6. Design of longitudinal reinforcement (Section 1.9)

To show that the longitudinal reinforcement is adequate, we must demonstrate that at each section the beam can resist a factored moment equal to $M_u + \frac{1}{2} d_v \Delta N_u$ where ΔN_u is the equivalent axial tension produced by the shear (Section 1.9.3). The calculations for these equivalent additional factored moments and for the nominal flexural capacities along the length of the beam are shown in Table 1.

The resulting required values of M_n are compared with the values of M_n provided by the prestressing strand in Fig. 27. It can be seen that the longitudinal reinforcement is adequate except near the end support.

To cover the 2000 kip-in. moment deficiency near the support, additional reinforcing bars welded to an anchor plate will be provided. If 3-#6 Grade 40 bars are used the additional moment capacity provided will be approximately:

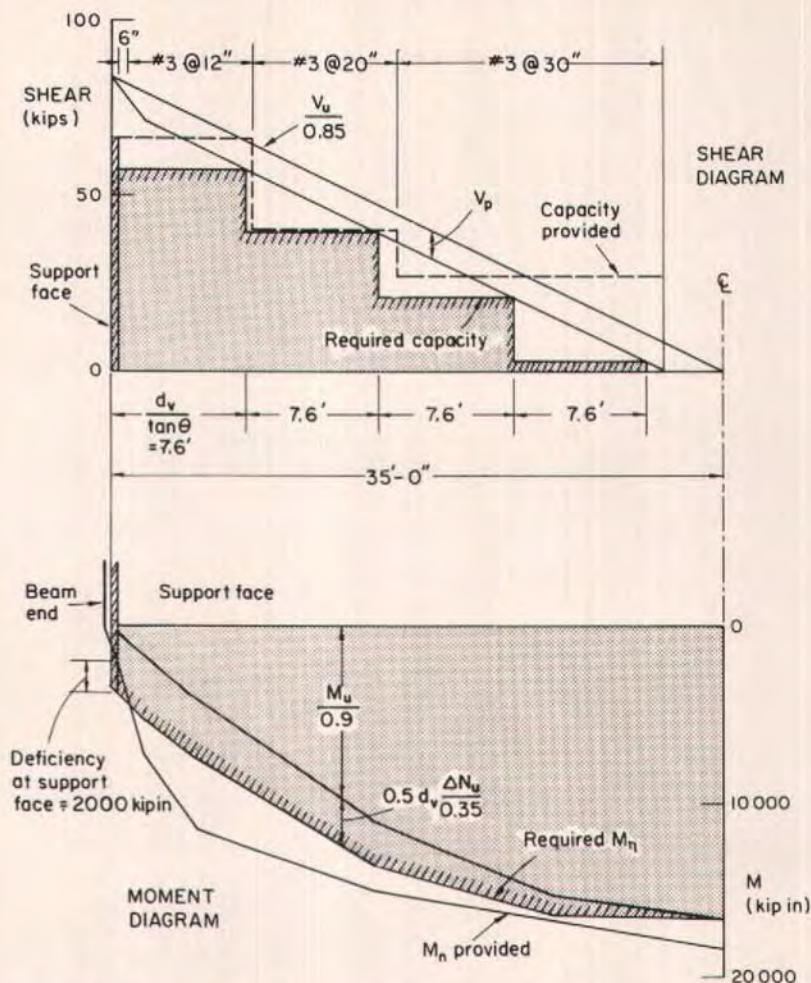


Fig. 27. Design Example 1—Shear and moment diagrams for design of stirrups and longitudinal reinforcement.

$$3 \times 0.44 \times 40 \times 37.5 = 1980 \text{ kip-in.}$$

If the length of the bearing pad is too small, then additional well anchored longitudinal steel spread some distance up the web will have to be provided to avoid local crushing of the concrete (Section 1.9.5.3). Check bearing pad length by Eq. (1-13) such that at the inner edge of the bearing area:

$$\frac{V_n}{b_v d_{ve}} \leq 0.012 (\theta - 10) f'_c$$

$$\frac{70.5}{(0.85 \times 8) d_{ve}} < 0.012 (20 - 10) 5$$

Therefore, $d_{ve} \geq 17.3$ in.

The 7 in. long bearing pad will provide an effective shear depth of:

$$l_b / \tan \theta = 7 / \tan 20 = 19.2 \text{ in.}$$

which is satisfactory.

Table 1. Additional factored moments required to be resisted and nominal moment capacities available.

Distance from end of beam	7 in.	25 in.	5 ft 3½ in.	15 ft 3½ in.	25 ft 3½ in.	35 ft 3½ in.
d (in.)	24.6	25.1	26.2	29.3	32.5	35.5
f_{ps} [Fig. 3.9.9, <i>PCI Design Handbook</i> and Eq. (18-3), ACI 318-77] (ksi)	42.0	150	227	264	264	265
$\beta_v = \frac{f_{ps}}{f_{ps} + E_s \epsilon_t} = \frac{f_{ps}}{0.7 \times 270 + 40} \leq 1.0$ (Section 1.9.1.2)	1.0	1.0	1.0	1.15	1.15	1.16
V_u (kips)	70.6	67.6	61.1	40.7	20.4	0
θ (degrees) (value chosen)	20	20	20	20	20	20
$\Delta N_u = \frac{\beta_v V_u}{\tan \theta}$ [Eq. (1-12)] (kips)	194	186	168	129	64	0
$\frac{1}{2} d_v \Delta N_u$ (Section 1.9.3) (kip-in.)	3200	3070	2770	2130	1060	0
$M_n = A_{ps} f_{ps} (d - a/2) = 13 \times 0.153 f_{ps} \left(d - \frac{13 \times 0.153 \times f_{ps}}{2 \times 0.85 \times 4 \times 96} \right)$ (kip-in.)	2040	7350	11520	14960	16640	18290

Note: 1 ft = 0.305 m; 1 in. = 25.4 mm; 1 ksi = 6.895 MPa; 1 kip-in. = 113 N · m

Table 2. Comparison of web reinforcement requirements by compression field theory and ACI 318-77 methods.

Design method	Location			
	Near support	About 12 ft from support	About 24 ft from support	Near midspan
Compression field theory	#3 @ 12 in.	#3 @ 20 in.	#3 @ 30 in.	0
ACI 318-77	#3 @ 20 in.	#3 @ 11 in.	#3 @ 20 in.	0

Note: 1 in. = 25.4 mm.

7. Comparison with PCI Design Handbook results

It is of interest to compare the results given above with the solution to the same design problem given by the

PCI Design Handbook.¹³ The Handbook uses the shear provisions of ACI 318-77. Table 2 compares the web reinforcement required by the two approaches.

DESIGN EXAMPLE 2—Parking Garage Spandrel Beam

The typical precast, prestressed concrete parking garage spandrel beam shown in Fig. 28 spans between column corbels. The double-tee floor framing system spans 60 ft, carries a live load of 50 psf, and is supported on the ledge of the spandrel beam.

1. Determine factored loads on the spandrel

Dead load of beam:

$$1.4 \times 0.725 = 1.02 \text{ kips/ft}$$

Dead load of deck:

$$1.4 \times 0.0895 \times 60/2 \times 4 = 15.04 \text{ kips per stem}$$

Live load:

$$1.7 \times 0.050 \times 30 \times 4 = 10.2 \text{ kips per stem}$$

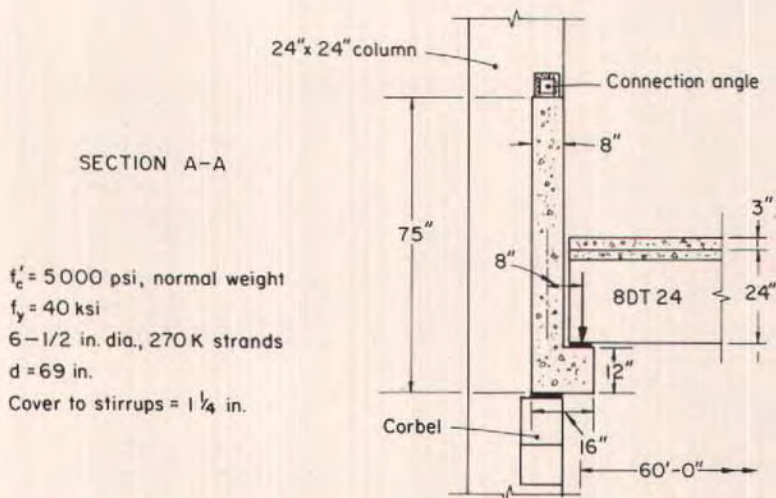
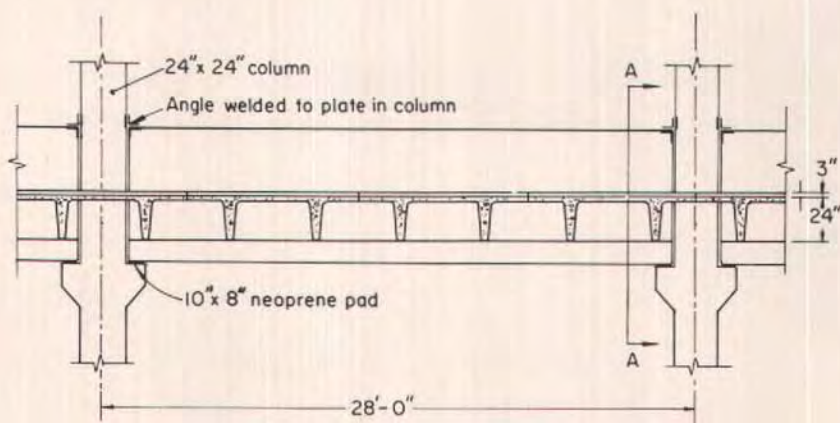


Fig. 28. Design Example 2—Structural framing of a precast prestressed concrete parking garage spandrel beam.

The loads and the resulting values of M_u , V_u , and T_u , are shown in Fig. 29.

2. Check minimum reinforcement requirements (Section 1.3)

To check Section 1.3.3, calculate the cracking moment under combined

loading M_{cr} at midspan. Assume effective prestress equals 150 ksi. Therefore, prestressing force:

$$P = 6 \times 0.153 \times 150 = 138 \text{ kips}$$

Therefore, bottom stress due to prestress:

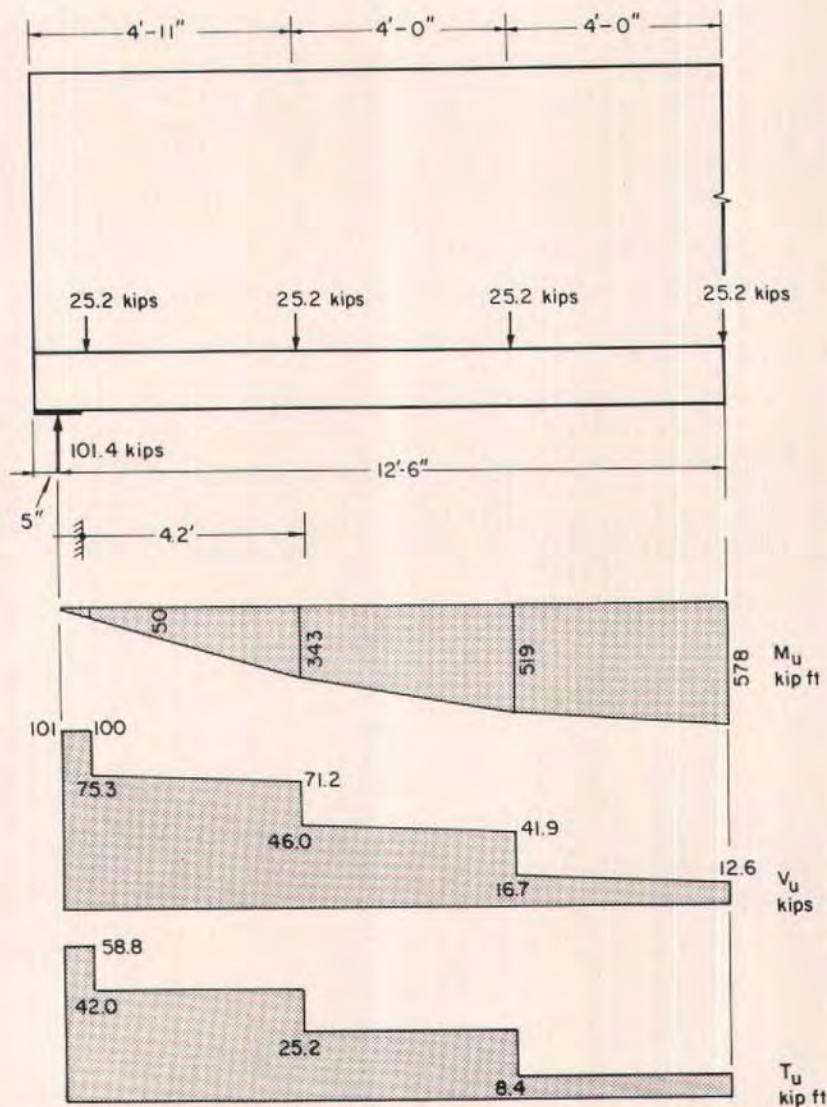


Fig. 29. Design Example 2—Moment shear, and torsion diagrams for spandrel beam.

$$\begin{aligned}
 f_{pe} &= \frac{P}{A} + \frac{Pe}{Z_b} \\
 &= \frac{138}{696} + \frac{138 \times 27.2}{10990} \\
 &= 0.540 \text{ ksi}
 \end{aligned}$$

Therefore, cracking moment:

$$\begin{aligned}
 M_{cr} &= Z_b (7.5 \sqrt{f'_c} + f_{pe}) \\
 &= (10,990/12) (0.530 + 0.540) \\
 &= 980 \text{ kip-ft}
 \end{aligned}$$

Since the maximum value of M_u (578 kip-ft) is considerably smaller than $1.2 M_{cr}$, we will use Section 1.3.2 and satisfy the minimum reinforcing requirements by increasing all design loads by one-third.

3. Check cross-sectional size and choose angle θ (Section 1.6)

We wish to check crushing of the concrete in the web of the beam at the face of the support. The load which is applied by the stem which is almost directly over the corbel can be transferred through the ledge of the beam to the support without causing shear stresses in the web of the spandrel beam. Hence:

$$\begin{aligned}
 V_u &= 75.3 \times 4/3 = 100.4 \text{ kips} \\
 \therefore V_n &= 100.4/0.85 = 118.1 \text{ kips} \\
 T_u &= 42.0 \times 12 \times 4/3 = 672 \\
 &\quad \text{kip-in.} \\
 \therefore T_n &= 672/0.85 = 791 \text{ kip-in.} \\
 b_v &= 8 - 2 \times 1.5 = 5.0 \text{ in.} \\
 &\quad (\text{unrestrained cover neglected} \\
 &\quad \text{and \#4 stirrups assumed}) \\
 d_v &= 75 - 2 \times 2 = 71 \text{ in.} \\
 &\quad (2 \text{ in. assumed from outside} \\
 &\quad \text{concrete surface to center of} \\
 &\quad \text{corner reinforcement}) \\
 A_{oh} &= 5 \times 72 + 8 \times 9 = 432 \text{ in.}^2 \\
 p_h &= 5 + 72 + 13 + 9 + 8 + 63 \\
 &= 170 \text{ in.}
 \end{aligned}$$

From Eq. (1-5), the nominal shear stress:

$$\begin{aligned}
 \tau_n &= \frac{V_n}{b_v d_v} + \frac{T_n p_h}{A_{oh}^2} \\
 &= \frac{118.1}{5 \times 71} + \frac{791 \times 170}{432^2} \\
 &= 1.053 \text{ ksi}
 \end{aligned}$$

$$\tau_n/f'_c = 1.053/5 = 0.211$$

For $f_y = 40$ ksi:

$$\epsilon_{tw} = 40/29,000 = 0.00138$$

Use $\epsilon_t = \epsilon_{tw}$.

From Eq. (1-6):

$$\begin{aligned}
 10 + \frac{35 (\tau_n/f'_c)}{0.42 - 50\epsilon_t} &< \theta \\
 &< 80 - \frac{35 (\tau_n/f'_c)}{0.42 - 65\epsilon_{tw}}
 \end{aligned}$$

$$\begin{aligned}
 10 + \frac{35 \times 0.211}{0.42 - 50 \times 0.00138} &< \theta \\
 &< 80 - \frac{35 \times 0.211}{0.42 - 65 \times 0.00138}
 \end{aligned}$$

$$31.0 \text{ deg} < \theta < 57.6 \text{ deg}$$

Therefore, section is adequate. Choose $\theta = 35$ deg.

4. Check diagonal crack control requirements (Section 1.7)

Since $f_y = 40$ ksi, the requirements of Section 1.7 will be satisfied by restricting the spacing of both the longitudinal and transverse reinforcement to 12 in. This spacing restriction could be relaxed for regions of the beam where calculations indicate that the cracking load exceeds the service load.

5. Design of transverse reinforcement (Section 1.8)

Design values of V_u and T_u near supports (Section 1.8.14) to be calculated at $d_v/(2 \tan \theta)$ from support face.

$$\frac{d_v}{2 \tan \theta} = \frac{71}{2 \tan 35} = 50.7 \text{ in. (or 4.2 ft)}$$

$$V_u = 45.9 \times 4/3 = 61.2 \text{ kips}$$

$$\therefore V_n = 61.2/0.85 = 72.0 \text{ kips}$$

$$T_u = 25.2 \times 12 \times 4/3 = 403 \text{ kip-in.}$$

$$\therefore T_n = 403/0.85 = 474 \text{ kip-in.}$$

Calculate torsional depth of compression, a_o , by Eq. (1-11):

$$a_o = \frac{A_{oh}}{p_h} \left[1 - \sqrt{1 - \frac{T_n p_h}{0.85 f'_c A_{oh}^2} \left(\tan\theta + \frac{1}{\tan\theta} \right)} \right]$$

$$= \frac{432}{170} \left[1 - \sqrt{1 - \frac{474 \times 170}{0.85 \times 5 \times 432^2} \left(\tan 35^\circ + \frac{1}{\tan 35^\circ} \right)} \right]$$

$$= 0.29 \text{ in.}$$

$$A_o = A_{oh} - \frac{1}{2} a_o p_h$$

$$= 432 - \frac{1}{2} \times 0.29 \times 170$$

$$= 407 \text{ in.}^2$$

6. Design of longitudinal reinforcement (Section 1.9)

The longitudinal steel requirements will be satisfied if the member is capable of resisting the positive factored moments given by $M_u + \frac{1}{2} d_v \Delta N_u$ and the negative factored moments given by $\frac{1}{2} d_v \Delta N_u - M_u$ (Section 1.9.3). However, in the end regions (within a distance $d_v/\tan\theta = 71/\tan 35^\circ = 8.45 \text{ ft}$), the negative moment that is to be resisted need not exceed:

$$\frac{d_v}{2} \frac{\beta_v}{\tan\theta} \frac{T_u p_o}{2 A_o} - M_u \text{ (Section 1.9.5.2)}$$

The resulting required moment capacities are shown in Fig. 30.

Table 3 summarizes the calculations of required additional factored moment for torsion and shear.

The longitudinal reinforcement that is continued along the entire length of the beam consists of 15—#4 bars and the 6— $\frac{1}{2}$ -in. diameter prestressing strands (see Fig. 32). This reinforcement results in the flexural capacities shown in Fig. 30. These capacities decrease near the ends of the beam because in these regions the strands are not fully developed.

From Fig. 30 it can be seen that both top and bottom additional longitudinal steel is required near the

Calculate transverse reinforcement for torsion by Eq. (1-10):

$$\frac{A_t}{s} = \frac{T_n \tan\theta}{2 A_o f_v}$$

$$= \frac{474 \times \tan 35^\circ}{2 \times 407 \times 40}$$

$$= 0.0102 \text{ in.}^2/\text{in.}$$

Calculate transverse reinforcement for shear by Eq. (1-8):

$$\frac{A_v}{s} = \frac{V_n \tan\theta}{d_v f_v}$$

$$= \frac{72.0 \times \tan 35^\circ}{71 \times 40}$$

$$= 0.0178 \text{ in.}^2/\text{in.}$$

Thus,

$$\frac{A_v}{s} + 2 \frac{A_t}{s} = 0.0381 \text{ in.}^2/\text{in.}$$

(for two legs)

Therefore, use #4 closed stirrups at 10-in. centers.

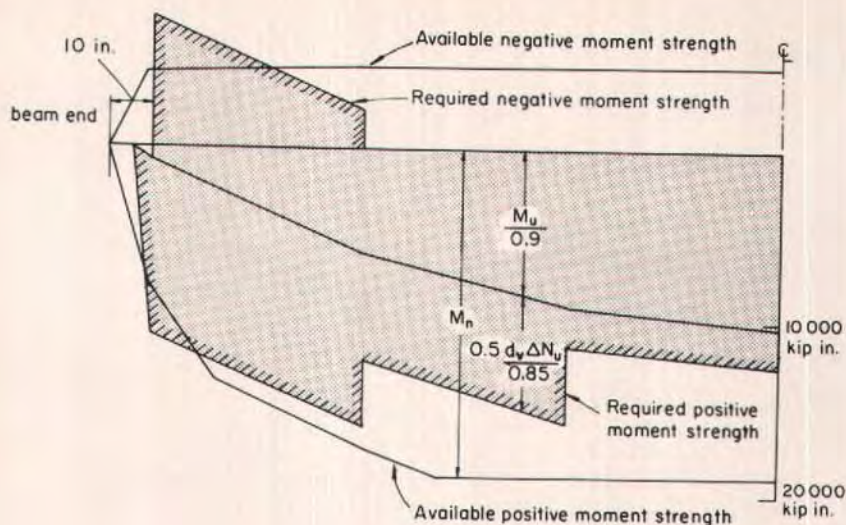


Fig. 30. Design Example 2—Required moment capacities along length of spandrel beam.

Table 3. Summary of calculations giving required additional factored moment for torsion and shear.

Distance from end of beam	10 in.	4 ft 11 in.	8 ft 11 in.	12 ft 11 in.	Units
f_{ps} [Fig. 3.9.9, <i>PCI Design Handbook</i> and Eq. (18-3), ACI 318-77]	60	218	258	258	ksi
$\beta_v = \frac{f_{ps}}{f_{ps} + E_s \epsilon_t} = \frac{f_{ps}}{0.7 \times 270 + 40}$ ≤ 1.0 (Section 1.9.1.2)	1.0	1.0	1.13	1.13	—
$V_u \times 4/3$	100.4	94.9/61.3*	55.9/22.3*	16.8	kips
$T_u \times 4/3$	672	672/403	403/134	134	kip-in.
θ	35	35	35	35	deg.
$\Delta N_u = \frac{\beta_v}{\tan \theta} \sqrt{V_u^2 + \left(\frac{T_u p_o}{2A_o}\right)^2}$ [Eq. (1-12)] $p_o = p_h - 4a_o = 170 - 4 \times 0.29$ $= 169$ in. $A_o = 407$ in. ²	245	241/148	162/58	52	kips
$\frac{1}{2} d_v \Delta N_u$ (Section 1.9.3)	8700	8555/5250	5750/2060	1850	kip-in.

*The two values refer to the sections either side of the applied stem load.

Note: 1 ft = 0.305 m; 1 in. = 25.4 mm; 1 in.² = 645.16 mm²; 1 kip = 4.448 kN; 1 kip-in. = 113 N • m; 1 ksi = 6.895 MPa.

ends of the member. The additional bottom steel must provide an additional moment capacity of 2720 kip-in. and the additional top steel must provide 3320 kip-in. If 3-#6 Grade 40 bars are welded to corner angles at the top and at the bottom the resulting additional moment capacity will be approximately:

$$3 \times 0.44 \times 40 \times 70 = 3695 \text{ kip-in.}$$

Use 4-ft long bars to cover the moment diagram.

Check end anchorage details of longitudinal steel (Section 1.9.5.3), Eq. (1-13):

$$\frac{V_n}{b_v d_{ve}} + \frac{T_n p_h}{A_{oh}^2} \leq 0.012 (\theta - 10) f'_c$$

$$\frac{100.4}{(0.85 \times 8) d_{ve}} + \frac{672 \times 170}{0.85 \times 432^2}$$

$$\leq 0.012 (35 - 10) 5$$

Therefore, $d_{ve} \geq 18.9$ in.

The 10 in. long bearing pad will provide an effective shear depth of $l_b / \tan \theta = 10 / \tan 35 = 14.3$ in. Therefore, an additional 4.6 in. of effective depth is required. Hence, provide a 5 x 5-in. corner angle to anchor the 3-#6 welded bars.

Note that in checking web crushing at the face of the bearing, it was assumed that the bearing pad and the ledge would prevent spalling of the cover; hence $b_p = 8$ in.

7. Check reinforcement details (Section 1.10)

Required diameter of longitudinal bars in corners of stirrups:

$$s \tan \theta / 16 = 10 \times \tan 35 / 16 = 0.44 \text{ in.}$$

Therefore, #4 corner bars are satisfactory.

Maximum spacing of shear reinforcement:

$$d_v / (3 \tan \theta) = 71 / (3 \tan 35) = 33.8 \text{ in.}$$

Maximum spacing of torsion reinforcement:

$$p_h / (8 \tan \theta) = 170 / (8 \tan 35) = 30.4 \text{ in.}$$

Therefore, spacing of bars at 10 in. is satisfactory.

8. Transfer of forces from ledge

It will be assumed that the load from the stem of the tee is transferred from the ledge to the bottom of the web of the spandrel by means of the compression strut shown in Fig. 31. This strut will be inclined at an angle θ , where $\tan \theta = 9/8$ (i.e., $\theta = 48.4$ deg).

Crushing of the concrete can be checked by Eq. (1-13) with $b_v = 3\frac{3}{4}$ in. and $d_{ve} = 1.5 + 6 / \tan 48.4 = 6.8$ in.

$$\frac{V_n}{b_v d_{ve}} \leq 0.012 (80 - \theta) f'_c$$

$$\frac{33.6}{0.85 \times 3.75 \times 6.8} \leq 0.012 (80 - 48.4) 5$$

$$1.55 \text{ ksi} < 1.90 \text{ ksi } (\therefore \text{ok})$$

Tension tie force required:

$$\frac{33.6}{0.9} \times \frac{8}{0.9 (9 + 1.5)} = 31.6 \text{ kips}$$

Area of steel required:

$$31.6 / 40 = 0.79 \text{ in.}^2$$

Therefore, use 4-#4 closed stirrups in ledge near each stem.

Calculate additional stirrups in web required to "hang-up" the load from the ledge (Section 1.8.17).

Reinforcement must be capable of

Table 4. Comparison of shear and torsion reinforcement by various methods.

Design method	Web reinforcement	Longitudinal reinforcement
Compression field theory	#4 at 10 in.	15—#4
<i>PCI Design Handbook</i>	#3 at 12 in.	14—#3
Zia and Hsu	#4 at 9 in.	12—#4

Note: 1 in. = 25.4 mm.

transmitting in tension a load of:

$$\frac{33.6}{0.9} \times \frac{75 - 12}{75} = 31.4 \text{ kips}$$

Required area of steel:

$$31.4/40 = 0.78 \text{ in.}^2$$

Therefore, at each stem location provide two double legged #4 closed stirrups in addition to the shear and torsion reinforcement.

The resulting reinforcement is shown in Fig. 32.

9. Comparisons with results of other design methods

It is of interest to compare the solution given above with solutions to essentially the same design problem given by the *PCI Design Handbook* and by Zia and Hsu.³⁴ Table 4 compares the shear and torsion reinforcement calculated by the three different approaches.

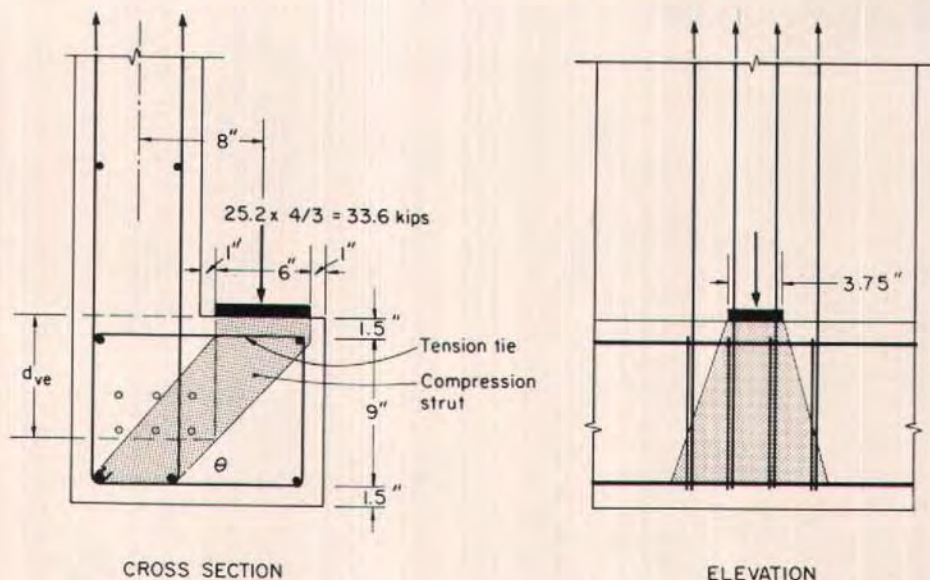


Fig. 31. Design Example 2—Transfer of forces from ledge.

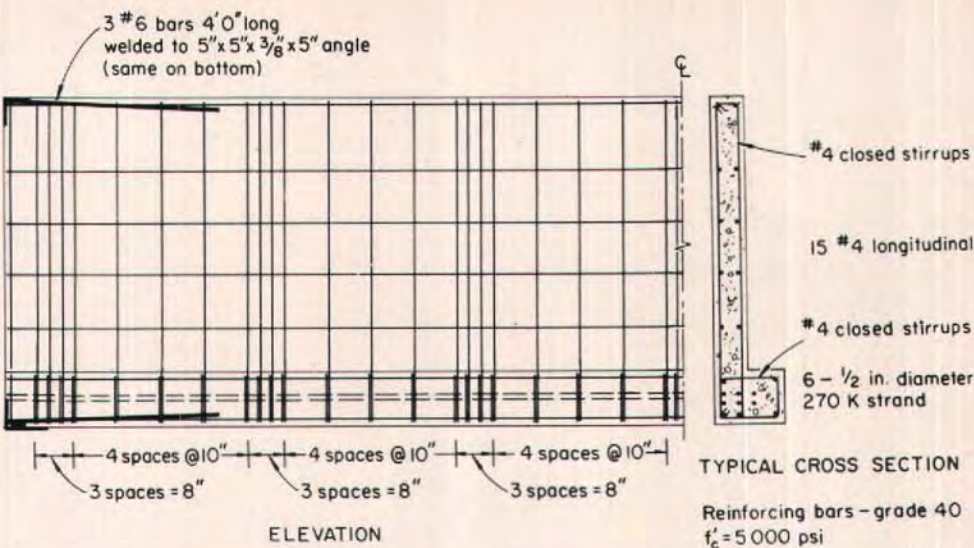


Fig. 32. Design Example 2—Spandrel beam reinforcement details.

* * *

DESIGN EXAMPLE 3—Precast Bridge Girder

The composite beam and slab two lane highway bridge described in Fig. 33 is to be designed to resist HS 20-44 loading.³⁵ The design of an interior girder will be described.

1. Determine loads on the girder

Dead Loads:

The dead loads are determined as shown in the accompanying table.

Live Loads:

Fraction of wheel load applied to

Determination of dead loads.

Dead loads supported by naked girder:

Girder	$685 \times 150/144$	= 0.714 kips/ft
Slab	$12 \times 8.20 \times 7.5$	= 738
Haunch	22×3	= 66
			$804 \times 150/144 = 0.838$ kips/ft

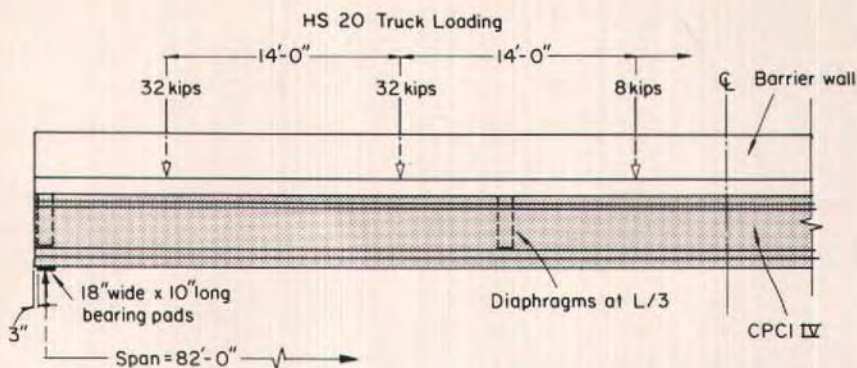
Diaphragms at 1/3 points

$$0.75 \times 3.66 \times 7.5 \times 0.150 = 3.09 \text{ kips}$$

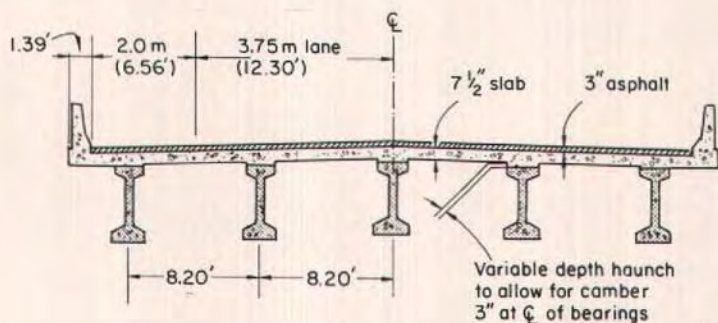
Dead loads supported by composite section:

3 in. asphalt	$37.72 \times 0.25 \times 150$	= 1415	lbs/ft
Barrier walls	$322.5 \times 2 \times 150/144$	= 672	lbs/ft
Rails (approx.)		= 48	lbs/ft

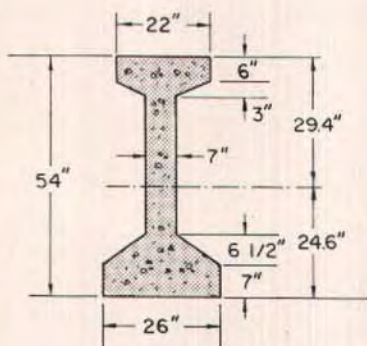
$$\therefore \text{Each composite girder takes } 2135/5 = 0.427 \text{ kips/ft}$$



ELEVATION



CROSS SECTION



CPCI IV GIRDER

MATERIALS:

Concrete

girders $f'_c = 6000$ psi

slab $f'_c = 4000$ psi

Strand

1/2" Φ 270 K low relaxation

Rebars

$f_y = 60$ ksi

Fig. 33. Design Example 3—Composite beam slab highway bridge.

each girder determined from Table 1.3.1(B) of AASHTO-1977.³⁵

$$\frac{\text{Girder spacing in ft}}{5.5} = \frac{8.2}{5.5} = 1.491$$

Therefore, fraction of truck load applied to each girder:

$$\frac{1}{2} \times 1.491 = 0.745$$

Impact allowance for truck load moments from AASHTO Section 1.2.12:

$$\begin{aligned} I &= 50/(L + 125) \\ &= 50/(82 + 125) \\ &= 24.2 \text{ percent} \end{aligned}$$

Note that for shear, I is a function of position along the span.

The maximum moments and shears in the girder which result from the application of the HS 20 loading are shown in Table 5.

2. Design prestressing (Fig. 34) and determine cracking loads

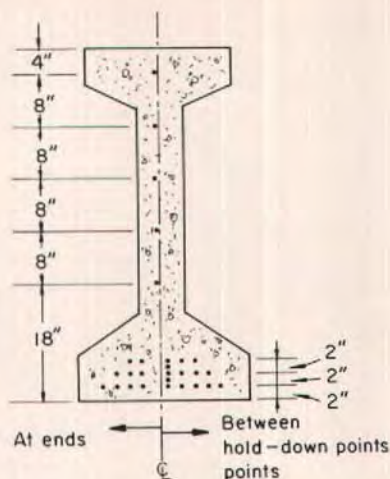


Fig. 34. Design Example 3—Strand pattern in CPCI IV Girders.

Based on the allowable stresses at transfer and at service loads design the required prestressing. The full details of these standard calculations will not be given. It was found, however, that if the stress in strands be-

Table 5. Calculated moments and shears for bridge girder.

Distance from support		0	0.1L	0.2L	0.3L	0.4L	0.5L
Service Loads M_s kip-ft	DL on naked girder	0	495	886	1172	1336	1388
	DL on composite	0	129	230	301	345	359
	Total DL	0	624	1116	1473	1681	1747
	LL + I	0	430	750	961	1063	1107
V_{se} kips	Total DL	84.2	68.0	51.7	35.5	16.2	0
	I, percent	24.2	25.2	26.2	27.4	28.7	30
	LL + I	59.0	52.8	46.5	40.1	33.6	27.0
Factored Loads M_u kip-ft	$1.3[D + \frac{5}{3}(L + I)]$	0	1743	3076	3997	4488	4670
	V_u kips	$1.3[D + \frac{5}{3}(L + I)]$	237	203	168	133	94

Note: 1 kip = 4.448 kN; 1 kip-ft = 1.356 kN · m.

fore transfer (f_{po}) was 204 ksi, then the stress in the strands after all losses (f_{se}) would be 164 ksi. Using hold-down points at the third points of the span, 28 - 1/2-in. diameter strands in the pattern described in Fig. 34 were chosen.

The stresses due to the chosen prestressing and the resulting calculated cracking loads are shown in Table 6. To simplify the calculations, the depth of the haunch which actually varies along the span was assumed constant at 1.5 in.

By comparing the calculated cracking shears, V_{cr} , with the service load shears, V_{se} ($D + L + I$), and recalling

that V_{cr} need not be taken less than $2\sqrt{f'_c} b_w d$ (Section 1.4.6) which is about 64 kips, it can be seen that diagonal cracks will not occur at service loads; hence the requirements of Section 1.7 will be satisfied.

To satisfy the minimum reinforcement requirements of Section 1.3.3, it is necessary that the nominal sectional strength V_n is at least equal to $1.2 V_{cr}$. For our girder this requirement will govern only for the section $0.1 L$ from the support.

3. Design transverse and longitudinal reinforcement

The design calculations for the

Table 6. Stresses due to prestress and cracking loads for bridge girder.

Distance from support centerline	5 in.*	0.1 L	0.2 L	0.3 L	0.4 L	0.5 L
Strand stress f_{se} (ksi)	85	164	164	164	164	164
Prestress force, P (kips)	364	703	703	703	703	703
e = eccentricity of P (in.)	10.6	13.6	16.7	19.9	20.9	20.9
Vertical component, V_p (kips)	11.6	22.4	22.4	22.4	0	0
Concrete stress, f_{pc} (ksi)	0.531	1.026	1.026	1.026	1.026	1.026
Concrete stress, f_{pe} (ksi)	0.923	1.996	2.217	2.445	2.516	2.516
Tensile stress due to DL on naked girder, f_{d0} (ksi)	0	0.602	1.078	1.426	1.626	1.689
Additional moment to crack = $Z_b \dagger (7.5 \sqrt{f'_c} + f_{pe} - f_{d0})$ (kip-ft)	1935	2542	2214	2059	1893	1812
\therefore Total moment to crack, M_{ocr} (kip-ft)	1935	3037	3100	3231	3229	3200
$V_{ocr} \ddagger$ Eq. (1-3) (kips)	187	257	271	285	267	267
$V_{cr} = 1/\sqrt{\left(\frac{1}{V_{ocr}}\right)^2 + \left(\frac{M_u/V_u}{M_{ocr}}\right)^2}$ (kips)	174	208	144	101	66	40
$\geq V_u/0.9\S$						
$V_n \geq 1.2 V_{cr}$ (kips)	263	250	187	148	104	66

* Taken at inner edge of bearing, 13 in. from end of beam.

† Z_b for composite section taken as 15440 in.³

‡ d taken as $7\frac{1}{2} + 1\frac{1}{2} + 29.4 + e$ in.

§ $\phi = 0.9$ for shear and 1.0 for flexure. AASHTO-77, Clause 1.6.5.

Note: 1 in. = 25.4 mm; 1 kip = 4.448 kN; 1 kip-ft = 1.356 kN · m; 1 ksi = 6.895 MPa.

transverse and longitudinal steel are summarized in Table 7. In determining d_p , it was assumed that the transverse steel would be anchored 2 in. above the bottom face of the girder and 5 in. above the top face of the girder. The 5-in. extension above the girder plus a standard hook will enable the #3 transverse bars to be fully developed at the interface. The contact surface at the interface is assumed to be clean and intentionally roughened. The transverse steel chosen to satisfy the design requirements is shown in Fig. 35.

With regard to longitudinal reinforcement, it can be seen from Table 7 that the prestressing strands alone are adequate ($1471 \approx 1475$) to resist the applied moment in addition to the equivalent moment caused by the shear.

Check crushing of web near support (Section 1.9.5.3).

$$\frac{V_n - V_p}{b_v d_{ve}} \leq 0.012 (\theta - 10) f'_c$$

$$\frac{251}{7 \times d_{ve}} \leq 0.012 (22 - 10) 6$$

Table 7. Design of transverse and longitudinal reinforcement for bridge girder.

Distance from support	0	0.1L	0.2L	0.3L	0.4L	0.5L
$V_n - V_p$ (kips)	251	228	165	126	104	66
$\frac{\tau_n}{f'_c} = \frac{V_n - V_p}{b_v s d_v f'_c} = \frac{V_n - V_p}{7 \times 57 \times 6}$	0.105	0.095	0.069	0.053	0.043	0.028
Range of θ Eq. (1-6) (deg)	21.6	20.5	17.6	15.9	14.8	13.0
$\epsilon_t = \epsilon_{tw} = 60/29000$ (deg)	$\rightarrow 67.1$	$\rightarrow 68.4$	$\rightarrow 71.5$	$\rightarrow 73.5$	$\rightarrow 74.7$	$\rightarrow 76.6$
Design choice of θ (deg)	22	22	22	22	22	22
$\frac{A_v}{s} = \frac{(V_n - V_p)}{d_v f_u} (\tan \theta)$ (in. ² /in.)	0.0297	0.0269	0.0195	0.0149	0.0123	0.0078
d (in.)	49.0	52.0	55.1	58.3	59.3	59.3
f_{ps} [ACI 318-77 12.10.1 + Eq. (18-3)] (ksi)	85	262	263	263	263	263
$\beta_v = \frac{f_{ps}}{204 + 60} < 1.0$	1.0	1.0	1.0	1.0	1.0	1.0
$\frac{\Delta N_u}{\phi} = \frac{\beta_v (V_n - V_p)}{\tan \theta}$ (kips)	621	564	408	312	257	163
$\frac{1}{2} d_v \frac{\Delta N_u}{\phi}$ (kip-ft)	1475	1340	969	741	610	388
$\frac{M_u}{\phi} + \frac{1}{2} d_v \frac{\Delta N_u}{\phi}$ (kip-ft)	1475	3083	4045	4738	5098	5058
$M_n = A_{ps} f_{ps} (d - a/2)$ $28 \times 0.153 f_{ps} \left(d - \frac{28 \times 0.153 \times f_{ps}}{2 \times 0.85 \times 4 \times 8.2 \times 12} \right)$ (kip-ft)	1471	4707	5015	5316	5410	5410

* $b_v = b_w$ since flanges restrain concrete cover.

Note: 1 in. = 25.4 mm; 1 kip = 4.448 kN; 1 kip-ft = 1.356 kN · m.

Table 8. Comparison of transverse reinforcement for bridge girder by various methods.

Various methods	Location	
	Near support	Near 0.25L
Compression field theory	#3 at 8 in.	#3 at 12 in.
ACI 318-77	#3 at 10 in.	#3 at 24 in.
AASHTO-77	#3 at 12 in.	#3 at 12 in.

Note: 1 in. = 25.4 mm.

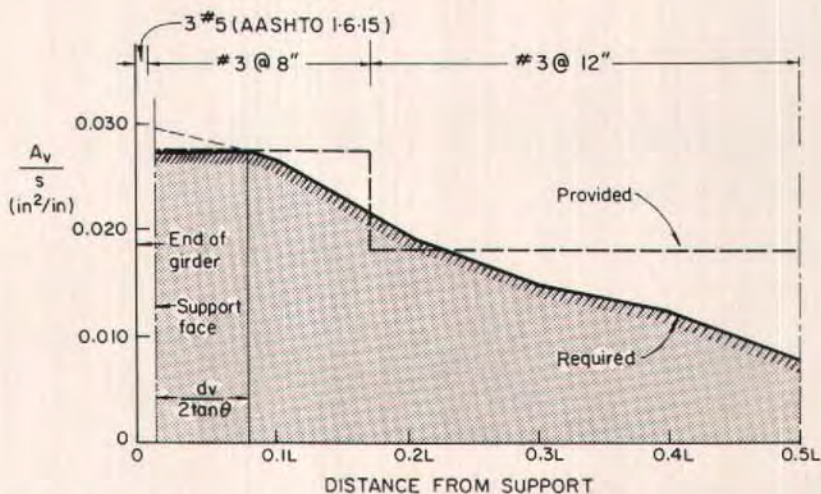


Fig. 35. Design Example 3—Design of transverse reinforcement.

from which

$$d_{ve} \geq 41.5 \text{ in.}$$

For this flanged section:

$$l_b = 10 + 13.5 = 23.5 \text{ in.}$$

$$\therefore d_{ve} = l_b / \tan \theta = 23.5 / \tan 22 = 58 \text{ in.,}$$

which exceeds the available depth above the junction of the web and flange.

Hence, available:

$$\begin{aligned} d_{ve} &= 54 + 5 - 13.5 \\ &= 45.5 \text{ in.} > 41.5 \text{ in. } (\therefore \text{ok}) \end{aligned}$$

4. Comparisons with other design methods

It is of interest to compare the transverse reinforcement obtained above with the transverse reinforcement that would have been obtained by satisfying the requirements of the ACI Code (with AASHTO load factors and ϕ factors) or by following the AASHTO recommendations (see Table 8).

* * *

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Design Example 3 was based on a worked example by Kris Bassi and Hid Grouni of the Ontario Ministry of Transportation and Communications which was presented at an Ontario Bridge Code Seminar in October, 1979.

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* * *

Metric (SI) Unit Equivalents

1 in.	= 25.4 mm
1 in. ²	= 645.16 mm ²
1 ft	= 0.3048 m
1 ft ²	= 0.0929 m ²
1 psf	= 47.88 N/m ²
1 psf	= 47.88 Pa
1 psi	= 0.006895 MPa
1 pcf	= 16.02 kg/m ³
1 kip	= 4.448 kN
1 kip/ft	= 14.594 kN/m
1 kip-in.	= 113 N • m
1 kip-ft	= 1.356 kN • m

NOTE: A References section and four appendices follow.

Discussion of this paper is invited. Please forward your comments to PCI Headquarters by May 1, 1981.

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APPENDIX A—DERIVATION OF STRESS BLOCK FACTORS

For the beam shown in Fig. 1, the resultant compression in the concrete is given by:

$$C = \int_0^c f_c b dy \quad (A1)$$

As the strain distribution is linear:

$$\frac{y}{c} = \frac{\epsilon_c}{\epsilon_{ct}} \quad (A2)$$

Using Eq. (A2) and assuming that the width of the compression zone is constant, Eq. (A1) can be rewritten as:

$$C = \frac{bc}{\epsilon_{ct}} \int_0^{\epsilon_{ct}} f_c d\epsilon_c \quad (A3)$$

If the equivalent uniform stress distribution is used, Fig. 1(d), then the resultant compression in the concrete is given by:

$$C = \alpha_1 \beta_1 b c f'_c \quad (A4)$$

Hence, for the magnitude of the resultant compression force to remain the same:

$$\alpha_1 \beta_1 = \frac{1}{f'_c \epsilon_{ct}} \int_0^{\epsilon_{ct}} f_c d\epsilon_c \quad (A5)$$

The position of the resultant concrete compression force can be determined by taking moments about the neutral axis. The distance up to the resultant force will be:

$$\bar{y} = \frac{\int_0^c f_c b y dy}{\int_0^c f_c b dy} \quad (A6)$$

Eliminating y by using Eq. (A2) leads to:

$$\bar{y} = \frac{\frac{b c^2}{\epsilon_{ct}^2} \int_0^{\epsilon_{ct}} f_c \epsilon_c d\epsilon_c}{\frac{bc}{\epsilon_{ct}} \int_0^{\epsilon_{ct}} f_c d\epsilon_c} \quad (A7)$$

For the equivalent uniform stress distribution, the distance from the neutral axis up to the resultant compression force will be:

$$\bar{y} = c (1 - \beta_1/2) \quad (A8)$$

Hence, for the position of the resultant compression force to remain the same:

$$(1 - \beta_1/2) = \frac{\int_0^{\epsilon_{ct}} f_c \epsilon_c d\epsilon_c}{\epsilon_{ct} \int_0^{\epsilon_{ct}} f_c d\epsilon_c} \quad (A9)$$

If the stress-strain curve of the concrete is known, then for a given value of maximum compressive strain, ϵ_{ct} , the stress block factors α_1 and β_1 can be determined from Eqs. (A9) and (A5).

If the following parabolic concrete stress-strain curve is assumed, see Fig. 1(e):

$$\frac{f_c}{f'_c} = 2 \frac{\epsilon_c}{\epsilon_o} - \left(\frac{\epsilon_c}{\epsilon_o} \right)^2 \quad (A10)$$

Then Eqs. (A9) and (A5) give:

$$\beta_1 = \frac{4 - \epsilon_{ct}/\epsilon_o}{6 - 2 \epsilon_{ct}/\epsilon_o} \quad (A11)$$

$$\alpha_1 \beta_1 = \frac{\epsilon_{ct}}{\epsilon_o} - \frac{1}{3} \left(\frac{\epsilon_{ct}}{\epsilon_o} \right)^2 \quad (A12)$$

Note: When using Eqs. (A11) and (A12) for beams in torsion replace ϵ_{ct} by ϵ_{dt} .

APPENDIX B—DERIVATION OF EXPRESSIONS FOR a_o , ϵ_t AND ϵ_l

For a beam in torsion the equivalent uniform concrete stress distribution consists of a uniform diagonal compressive stress of $\alpha_1 f'_c$ acting over a depth of a_o . These diagonal stresses acting at an inclination of θ will produce a shear flow [see Eq. (2)] of:

$$q = \alpha_1 f'_c a_o \sin\theta \cos\theta \quad (B1) \quad \text{and}$$

From Eq. (5) and Eq. (B1):

$$\frac{\Delta N}{\alpha_1 f'_c p_o} = a_o \frac{\sin\theta \cos\theta}{\tan\theta} \quad (B2)$$

From Eq. (6) and Eq. (B1):

$$\frac{A_t f_t}{\alpha_1 f'_c s} = a_o \sin\theta \cos\theta \tan\theta \quad (B3)$$

Adding Eqs. (B2) and (B3) gives:

$$\begin{aligned} \frac{\Delta N}{\alpha_1 f'_c p_o} + \frac{A_t f_t}{\alpha_1 f'_c s} \\ = a_o \sin\theta \cos\theta \left(\frac{1}{\tan\theta} + \tan\theta \right) = a_o \end{aligned} \quad (B4)$$

which completes the derivation of Eq. (10).

From Fig. 5 it can be seen that:

$$a_o = \beta_1 t_d = \beta_1 \frac{\epsilon_{ds}}{\phi_d} = \beta_1 \frac{\epsilon_{ds}}{\psi \sin 2\theta} \quad (B5)$$

where ψ is the twist of the beam. The twist can be determined²³ as:

$$\psi = \gamma_{tt} \frac{p_h}{2A_{oh}} \quad (B6)$$

where γ_{tt} is the shear strain between

longitudinal and transverse lines in the plane of the hoop centerline.

From Mohr's circle of strain (see Fig. B1):

$$\gamma_{tt} = \frac{2(\epsilon_{ds} + \epsilon_l)}{\tan\theta} \quad (B7)$$

$$\gamma_{tt} = 2(\epsilon_{ds} + \epsilon_l) \tan\theta \quad (B8)$$

[Note: eliminating γ_{tt} from these two equations results in Eq. (7)].

Substituting from Eqs. (B7) and (B8) into Eq. (B5) gives:

$$a_o = \beta_1 \frac{\epsilon_{ds} A_{oh}}{p_h (\epsilon_{ds} + \epsilon_l) \sin 2\theta} \quad (B9)$$

But from Eq. (B2):

$$\begin{aligned} a_o &= \frac{\Delta N}{\alpha_1 f'_c p_o} \frac{\tan\theta}{\sin\theta \cos\theta} \\ &= \frac{2 \Delta N}{\alpha_1 f'_c p_o} \frac{\tan\theta}{\sin 2\theta} \end{aligned} \quad (B10)$$

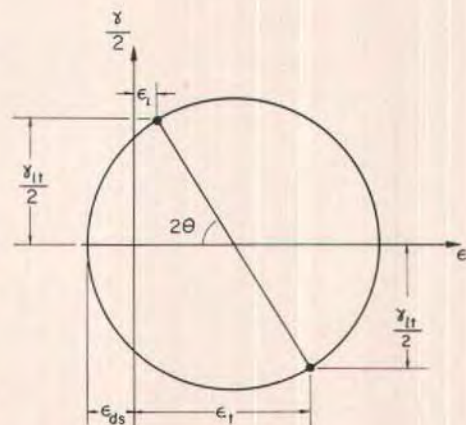


Fig. B1. Mohr's circle of strain.

Comparing Eqs. (B9) and (B10) we see that:

$$\beta_1 \frac{\epsilon_{ds} A_{oh}}{p_h (\epsilon_{ds} + \epsilon_t)} = \frac{2 \Delta N}{\alpha_1 f'_c p_o} \quad (\text{B11})$$

Rearranging this equation gives the tensile strain of longitudinal reinforcement due to shear and/or torsion:

$$\epsilon_t = \left[\frac{\alpha_1 \beta_1 f'_c A_{oh} p_o}{2 p_h \Delta N} - 1 \right] \epsilon_{ds} \quad (\text{B12})$$

which completes the derivation of Eq. (12). By substituting from Eqs. (B8) and (B6) into Eq. (B5) and comparing the resulting expression for a_o with Eq. (B3), Eq. (11) can be derived.

* * *

APPENDIX C—DERIVATION OF CRACK CONTROL LIMIT ON θ

We wish to find the lower limit on θ which will ensure that when $V = V_{se}$, the strain in the transverse steel does not exceed 0.001. Thus, from Eq. (30):

$$\left[1 - \left(\frac{V_{cr}}{V_{se}} \right)^3 \right] \epsilon_t \leq 0.001 \quad (\text{C1})$$

For a beam subjected to a shear V_{se} , the strain ϵ_t can be calculated from Eq. (3) as:

$$\epsilon_t = \frac{V_{se}}{E_s A_v d_v} \frac{s}{d_v} \tan \theta_s \quad (\text{C2})$$

where θ_s is the inclination of the diagonal stresses at the service load shear V_{se} .

The area of shear reinforcement A_v is determined from Eq. (3) by using:

$$A_v = \frac{V_n}{f_y} \frac{s}{d_v} \tan \theta \quad (\text{C3})$$

Substituting Eq. (C3) into Eq. (C2) gives:

$$\epsilon_t = \frac{f_y}{E_s} \frac{V_{se}}{V_n} \frac{\tan \theta_s}{\tan \theta} \quad (\text{C4})$$

In design we will choose θ and hence to find ϵ_t at service load we need a relationship between θ_s , the angle of inclination of the diagonals at service load, and θ , the angle of inclination of the diagonals at the nominal capacity.

In developing an approximate relationship between θ_s and θ we should recognize that the higher the estimate of θ_s the higher will be the estimate of ϵ_t [Eq. (C4)]. Hence, for our purposes overestimating θ_s will be conservative.

For a known value of V_n and a given choice of θ , the required areas of transverse and longitudinal reinforcement could be calculated. The response of the beam at service load could then be determined to find the value of θ_s . It will be found that for a given value of θ , the higher the value of V_n , the higher the value of θ_s . Hence, to determine a conservative value of θ_s for a given value of θ , the highest possible value of V_n should be used, that is, the value which just satisfies the concrete crushing limit of Eq. (23) should be used.

The calculated relationships between θ_s and θ for a non-prestressed and for a prestressed series of beams

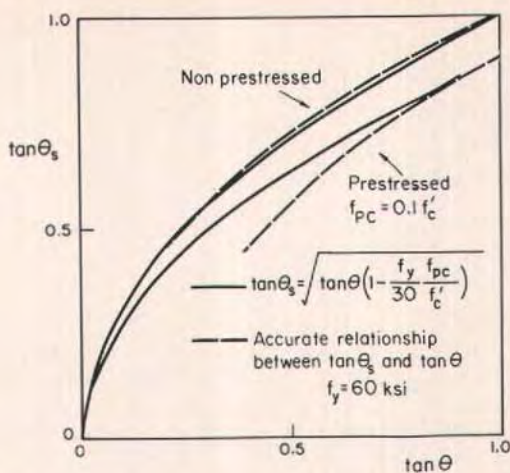


Fig. C1. Relation between θ_s and θ for prestressed and non-prestressed concrete beams.

are shown in Fig. C1. In calculating these relationships it was assumed that $V_{se}/V_n = 0.55$. Additionally, for the prestressed beams it was assumed that $f_{se} = 150$ ksi and $\Delta\epsilon_p = 6.5 \times 10^{-3}$. Also shown in Fig. C1 is the approximate relationship.

$$\tan\theta_s = \sqrt{\tan\theta \left(1 - \frac{f_y f_{pc}}{30 f'_c} \right)} \quad (C5)$$

It can be seen that this equation represents reasonably well the relationship between θ_s and θ .

Substituting from Eqs. (C5) and (C2) into Eq. (C1) results in Eq. (31):

$$\tan\theta \geq \left(\frac{f_y V_{se}}{29 V_n} \right)^2 \left(1 - \frac{f_y f_{pc}}{29 f'_c} \right) \times \left[1 - \left(\frac{V_{cr}}{V_{se}} \right)^3 \right]^2 \quad (31)$$

* * *

APPENDIX D—NOTATION

Symbols not defined in the notation section of the Design Recommendations are defined below:

A_t = area of longitudinal reinforcing bars
 A_p = area of longitudinal prestressing steel

a = equivalent depth of compression
 b = width of beam
 C = resultant compressive force in concrete
 c = neutral axis depth
 D = diagonal compressive force in concrete

d	= effective depth to flexural reinforcement	y	= distance from neutral axis
d_c	= distance from extreme compression fiber to neutral axis	y_t	= distance from centroidal axis to extreme fiber in tension
d_d	= duct diameter	\bar{y}	= distance from neutral axis to resultant compressive force
e	= eccentricity of prestressing	Z_b	= cross-sectional section modulus for bottom fiber
f_c	= compressive stress in concrete corresponding to strain ϵ_c	α_1	= stress block factor defined in Eq. (1)
f_{ct}	= concrete stress corresponding to the strain ϵ_{ct}	γ	= shear strain
f_d	= equivalent uniform compressive stress	γ_{lt}	= shear strain between longitudinal and transverse lines
f_{du}	= limit of principal diagonal compressive stress in diagonally cracked concrete	γ_m	= maximum shear strain
f_p	= stress in longitudinal prestressing steel	ϵ	= normal strain
f_{du}	= tensile stress due to dead load on naked girder (Example 3)	ϵ_c	= compressive strain in concrete
f_{py}	= "yield" stress of prestressing steel; taken as 0.2 percent offset stress	ϵ_{cp}	= strain in concrete at level of prestressing steel
f_{se}	= effective stress in prestressed reinforcement (after allowance for all prestress losses)	ϵ_{ct}	= maximum concrete compressive strain
f_t	= stress in hoop reinforcement	ϵ_d	= diagonal compressive strain
f_v	= stress in shear reinforcement	ϵ_{ds}	= concrete diagonal compressive strain at effective surface of beam in torsion
I	= impact fraction (Example 3)	ϵ_o	= strain in concrete corresponding to maximum compressive stress
jd	= lever arm for flexural resistance	ϵ_p	= strain in prestressing
L	= length of member	$\Delta\epsilon_p$	= difference in strain between prestressing steel and surrounding concrete
M	= flexural moment	ϵ_t	= strain in transverse reinforcement
M_s	= service load flexural moment	ϵ_{te}	= expected transverse strain at service load level
M_u	= factored flexural moment	θ	= angle of diagonal compressive stresses in concrete
ΔN	= tensile force in longitudinal steel produced by shear and torsion	θ_e	= value of θ at end of beam (sec. Fig. 21)
P	= prestressing force	θ_s	= value of θ at service load
q	= shear flow	ϕ	= curvature
T	= torsional moment	ϕ_d	= curvature of walls due to torsion
t_d	= effective wall thickness under torsional loading	ψ	= twist per unit length
V	= shear	σ	= normal stress
v_u	= nominal shear stress at ultimate	τ	= shear stress
w_u	= factored uniform load per unit length		

* * *