Fatigue Resistance of Reinforcement in Partially Prestressed Beams

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The ranges of stress in the prestressed and non-prestressed reinforcement of a variety of partially prestressed concrete beams are analyzed using a computer program based on the conventional elastic theory of reinforced concrete.

The results are compared with typical modified Goodman diagrams and lead to some general conclusions on the design of partially prestressed beams for resistance to fatigue.
The principle of partial or limited prestressing is now accepted in the CEB Recommendations and British Code CP 110:1972 by the provision for 'Class 3' structural members, that is, prestressed members in which controlled cracking at the tensile face is permitted.

The ACI Code (ACI 318-71) is less explicit, but Sections 18.4.2(c) and 18.4.3 clearly envisage flexural cracking.

In all countries, however, development in this area has been restricted by the lack of knowledge of the resistance of partially prestressed members to fatigue, and there is a need for simple and safe design criteria.

Most prestressed concrete members are under-reinforced and, if fatigue failure is possible, it is the performance of the tensile reinforcement, either prestressed or non-prestressed, which is likely to be critical.

A possible method for assessing the fatigue strength, originally proposed by Eckberg et al., and more recently applied to members with non-prestressed reinforcement, is to calculate the range of stress in each type of reinforcement when the member is subjected to a fluctuating load and to ascertain whether the range falls outside the fatigue limits defined by the modified Goodman diagram for the appropriate grade of steel.

Previous work has shown that the stresses in the reinforcement of partially prestressed beams may be calculated fairly accurately by modifying the conventional linear elastic theory, formerly used for reinforced concrete. The details of the calculation, which was programmed for a computer, are given in Appendix A.

The cross section of the beams which were analyzed, and the position of the post-tensioned bonded tendons and the non-prestressed supplementary reinforcement are shown in Fig. 1.

Three types of non-prestressed reinforcement were considered, namely, plain hot-rolled mild steel bars (in beams of Series M), deformed hot-rolled high-tensile bars (Series H), and plain cold-drawn wire (Series W).

In order to provide a basis of comparison, the prestressed reinforcement in all the beams was assumed to consist of plain cold-drawn wire conforming to

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**Method of Analysis**

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Fig. 1. Cross section of beams.
the modified Goodman diagram given by Ros⁴ (Fig. 2).

The bar reinforcement was assumed to have fatigue properties defined by the diagrams of Menzies (Fig. 3)⁵ in which it will be noted that the fatigue strength of the plain hot-rolled mild steel bar is greater than that of the deformed high-tensile bar notwithstanding its lower static tensile strength.

The ultimate tensile resistance of one 7 mm cold-drawn wire was approximately equal to that of two 10-mm high-tensile bars or three 10 mm mild steel bars; so that by using these sizes, it was possible to compare beams which had approximately equal static ultimate moments but different types and percentages of non-prestressed reinforcement.

The various combinations of reinforcement are given in Table 1, which also shows the design cracking moment and ultimate moment (see p. 85) calculated according to ACI 318-71.

Discussion of Results

The relationships between the moment and the reinforcement stresses are shown for each type of non-prestressed reinforcement in Figs. 4 through 6.

In each instance there is a slow increase of stress followed by a transition to a more rapid increase corresponding to the development of cracking in the concrete. On the initial loading, the first part of the curve continues to the cracking moment; but under repeated loading, the cracks open and close at a lower moment.

For this reason the tensile resistance of the concrete is neglected in the cal-

![Fig. 2. Modified Goodman diagram for fatigue of cold-drawn wire (Ros⁴).](image-url)
calculation and the transition, therefore, commences at the point of decompression of the concrete at the soffit of the beam.

After decompression the stress increases at about the same rate in the prestressed and non-prestressed reinforcement, and it is assumed to lie within the elastic limit over the range of service moment. The diagram shows the assumption to be justified with the exception of the stress in the prestressed reinforcement of Beam W4, and possibly also Beam W3.

The effect of prestress is obviously to delay the onset of cracking so that the transition occurs progressively earlier in numbers 1 through 4 of the beams in each series, as the number of tendons is reduced.

After cracking, however, the rate of increase of stress is greater in beams with a smaller total area of reinforcement; thus, the general rate of increase is greater in Series H (Fig. 5) than in Series M (Fig. 6), and greatest in Series W (Fig. 4).

In each series the increase of reinforcement stress will obviously be greater in Beams 5 through 7 which have a smaller area of reinforcement than Beams 1 through 4.

In some instances the greater increase of stress in the cracked condition offsets the advantage of later opening of the cracks as will be seen from a comparison of H3 with H4 in Fig. 5 or M2 with M4 in Fig. 6.

To assess the probable resistance of a beam to fatigue, it is necessary to

Fig. 3. Modified Goodman diagram for fatigue of bar reinforcement of $2 \times 10^6$ cycles (Menzies').
Fig. 4. Computed relationship between bending moment and reinforcement stresses (Series W Beams).

Fig. 5. Computed relationship between bending moment and reinforcement stresses (Series H Beams).
Fig. 6. Computed relationship between bending moment and reinforcement stresses (Series M Beams).

Fig. 7. Reinforcement stress ranges on modified Goodman diagrams (Series W Beams).
Fig. 8. Reinforcement stress ranges on modified Goodman diagrams (Series H Beams).

Fig. 9. Reinforcement stress ranges on modified Goodman diagrams (Series M Beams).
compare the ranges of stress in the prestressed and non-prestressed reinforcement under a fluctuating load with the modified Goodman diagram for the estimated number of cycles.

The ranges of service moment considered are those shown in Table 1 which represent equal dead and live loads each equal to about one-third of the ultimate load, slightly greater than would be permitted by the ACI requirement: \( U = 1.4D + 1.7L \).

The calculated stress ranges for each beam are shown as vertical lines on the modified Goodman diagrams in Figs. 7 through 9.

In Fig. 7 the ranges of stress in the cold-drawn wire used as non-prestressed reinforcement are seen to lie within the limits of the diagram; but in the prestressed reinforcement, the stress range crosses the boundary indicating a risk of fatigue failure in all the beams except W1 and W5 in which the whole of the reinforcement is prestressed.

It will be seen from Fig. 8 that where deformed high strength steel is used the range of stress is close to the limit except in Beam H2, but the ranges of stress in the prestressing steel are well within the limits for all the beams.

Fig. 9 indicates that when mild steel is used as non-prestressed reinforcement, there appears to be little risk of fatigue failure in any of the beams; the larger areas of reinforcement have reduced the stress ranges, while the upper stress limit for plain hot-rolled bars is higher than for deformed bars.

### Table 1. Details of beams.

<table>
<thead>
<tr>
<th>Beam Number</th>
<th>No. of Tendons</th>
<th>Series W No. of Wires*</th>
<th>Series H No. of Bars*</th>
<th>Series M No. of Bars*</th>
<th>Design Moment Cracking (kNm) (in.-kips)</th>
<th>Design Ultimate Moment (kNm) (in.-kips)</th>
<th>Range of Working Moment Investigated (kNm) (in.-kips)</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td></td>
<td>20 (177)</td>
</tr>
<tr>
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<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>30.0 (266)</td>
<td></td>
<td>58 (513)</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>22.8 (202)</td>
<td></td>
<td>40 (355)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>15.6 (138)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>30.0 (266)</td>
<td></td>
<td>15.5 (137)</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>22.8 (202)</td>
<td></td>
<td>45 (398)</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>15.6 (138)</td>
<td></td>
<td>31 (274)</td>
</tr>
</tbody>
</table>

* Non-prestressed reinforcement - Series W: 7 mm (0.28 in.) diameter cold drawn prestressing wire, \( f_{pu} = 1900 \text{ N/mm}^2 \) (277 ksi)
Series H: 10 mm (0.4 in.) diameter high strength deformed bars, \( f_y = 410 \text{ N/mm}^2 \) (60 ksi)
Series M: 10 mm (0.4 in.) diameter plain mild steel bars, \( f_y = 275 \text{ N/mm}^2 \) (40 ksi)

** Calculated according to ACI 318-71 for \( f_c' = 48 \text{ N/mm}^2 \) (7000 psi)

*** Calculated according to ACI 318-71 for \( f_c' = 48 \text{ N/mm}^2 \) (7000 psi) (According to CP110 for \( f_{cu} = 1.25 f_c' = 60 \text{ N/mm}^2 \) (8750 psi) values are up to 5 percent lower.)
Design Recommendations

In order to safeguard against fatigue failure under a repeated live load, it is necessary to limit the range of stress in the prestressed and non-prestressed reinforcement to a safe level.

This may be done in two possible ways:

1. By prestressing so that the decompression load is greater than the dead load, i.e., the concrete at the bottom of the beam is in compression under the dead load.

2. By using a sufficiently large area of non-prestressed reinforcement.

It would appear from Figs. 4 and 7 that only the first alternative is possible if cold-drawn prestressing wire is used as non-prestressed reinforcement; since W1 and W5, the only beams not liable to fatigue failure are those in which the concrete is in compression under dead load.

With hot-rolled deformed bars, this requirement is also advisable; for although the increase of reinforcement stress after decompression is reduced by the greater area of steel, the fatigue resistance of the bars is less (Figs. 5 and 8).

When mild steel is used as non-prestressed reinforcement, the area of reinforcement is greatly increased and the range of stress is relatively low even after decompression, as seen from Figs. 6 and 9.

Beams of this type should therefore be able to resist fatigue over the complete range from reinforced concrete to fully prestressed concrete; and although the use of low strength steel is generally not economical, it may in certain circumstances be advantageous when designing against fatigue.

References


2. Bennett, E. W., "Fatigue of Reinforcement in Beams with Limited Prestress," Abeles Symposium on Fatigue of Concrete, Special Publication SP-41, American Concrete Institute, 1974, pp. 301-313.


APPENDIX A—CALCULATION OF REINFORCEMENT STRESSES

Assumptions

The following assumptions were made in the analysis:

1. Both concrete and steel are elastic with a linear stress-strain response.

2. Plane sections remain plane at any stage of loading; and therefore, the change of stress and strain due to bending is proportional to the distance from the neutral surface.

3. Concrete cannot withstand any tensile stress.

4. The prestress in the top fiber of the concrete is zero.

5. There is perfect bond between steel and concrete.
The analysis of the beams prior to cracking is based on simple elastic theory assuming a homogenous concrete section. After flexural cracking, the strain and stress distributions are as shown in Fig. A.

From the equilibrium of forces:

\[
\rho_s b d (f_{cs} + \beta_s \Delta f_s) + \rho_i b d (\beta_i \Delta f_i - f_{is}) = \frac{1}{2} b x f_{c_{sup}}
\]  

where \( \beta_s \) and \( \beta_s \) are strain distribution factors given by:

\[
\beta_s = \frac{(d_a - x_b)/(d - x_b)}{x_b}
\]

\[
\beta_i = \frac{(d_i - x_b)/(d - x_b)}{x_b}
\]

From strain compatibility:

\[
\frac{(f_{c_{sup}}/E_c)(\Delta f_s/E_ps - f_{cs}/E_c)}{\xi/(1 - \xi)} = \xi
\]  

From the equilibrium of moments about the center of compression:

\[
M = \rho_s b d^2 f_{ps} (d_a/d - \xi/3) + \beta_s \rho_s b d^2 \Delta f_s (d_a/d - \xi/3) + \beta_i \rho_i b d^2 \Delta f_i (d_i/d - \xi/3) - \rho_i b d^2 f_{is} (d_i/d - \xi/3)
\]  

Eliminating \( f_{c_{sup}} \) between Eqs. (1) and (2) and substituting for \( \Delta f_s \) in Eq. (3), a cubic in \( \xi \) is derived.

\[
\xi(\psi f_{c_{sup}} + f) + 3\xi^2[M/bd^2 - (f_{psi} \Delta f_s + f)] + [M/bd^2 - (f - f_{psi}/\psi)] = 6\alpha \psi \xi - 6\alpha \psi = 0
\]

where

\[
a = \text{average modular ratio, i.e., } E_{ps}/E_c = (E_p + E_s)/2E_c
\]

\[
\phi = \beta_s \rho_s + \beta_i \rho_i
\]

\[
\phi' = \beta_i \rho_i d_i/d + \beta_s \rho_s d_s/d
\]

\[
f = \rho_i f_{ps} - \rho_s f_{is}
\]

\[
f' = \rho_s f_{ps} d_s/d - \rho_i f_{is} d_i/d
\]

Eq. (4) may be solved by any converging procedure, making initial approximations for \( \beta_s, \beta_i, \) and \( d \), which are then corrected by successive iteration.

Knowing \( \xi, \Delta f_s \) is easily determined from Eqs. (1) and (2) and hence the stresses in the reinforcement:

\[
f_p = \beta_s \Delta f_s E_p/E_{ps} + f_{ps}
\]

\[
f_s = \beta_i \Delta f_s E_s/E_{ps} - f_{is}
\]

The above solution is for a rectangular compression area and requires only slight modification to accommodate flanged sections.

The iterative procedures required are greatly facilitated by the use of a computer. A program was written for use with an ICL 1906A computer which allowed rapid determination of the reinforcement stresses at any value of applied moment, \( M \).
APPENDIX B—NOTATION

\[\begin{align*}
A_p & = \text{area of prestressing tendons} \\
A_s & = \text{area of non-tensioned tensile reinforcement} \\
b & = \text{breadth of section} \\
d & = \text{effective depth of center of tension} \\
d_p & = \text{effective depth of prestressing tendons} \\
d_s & = \text{effective depth of non-tensioned reinforcement} \\
E_e & = \text{secant modulus of elasticity of concrete} \\
E_r & = \text{modulus of elasticity of prestressing tendons} \\
E_s & = \text{modulus of elasticity of non-tensioned reinforcement} \\
E_{ps} & = \text{average modulus of elasticity of tensile reinforcement, i.e., } (E_p + E_s)/2 \\
f_{ps} & = \text{initial stress in concrete at average level of tendons} \\
f_{c,up} & = \text{stress in concrete at top face due to prestress and bending} \\
f_{pe} & = \text{effective prestress in tendons after losses due to prestress} \\
\Delta f_r & = \text{additional average stress in tensile reinforcement due to bending} \\
f_p & = \text{total stress in tendons due to prestress and bending} \\
f_s & = \text{total stress in non-tensioned reinforcement due to prestress and bending} \\
M & = \text{applied moment} \\
x & = \text{depth of neutral axis of stress} \\
a & = \text{average modular ratio, i.e., } (E_p + E_s)/2E_e \\
\xi & = \text{neutral axis depth coefficient, i.e., } x/d \\
\rho_p & = \text{ratio of area of prestressing tendons to concrete cross-sectional area, i.e., } A_p/bd \\
\rho_s & = \text{ratio of area of non-tensioned reinforcement to concrete cross-sectional area, i.e., } A_s/bd \\
f_{ps} & = \text{ultimate strength of prestressing tendon} \\
f_y & = \text{yield or proof stress of reinforcing bar}
\end{align*}\]

Discussion of this paper is invited. Please forward your comments to PCI Headquarters by Sept. 1, 1977.