The analysis of prestressed concrete sections at ultimate capacity is a simple procedure if one follows the ACI or the AASHTO specifications.

These specifications provide a number of simplifying assumptions, namely, that the state of strain in the concrete compressive zone at ultimate is known, that the force in the concrete can be approximated from the equivalent rectangular stress block and that the stress in the prestressing steel can be approximated from materials and section properties.

It has been shown that these assumptions lead to satisfactory predictions of ultimate moments for fully prestressed and somewhat partially prestressed normal weight concrete beams.

Partial prestressing implies the use in combination with prestressing steel of non-prestressed conventional reinforcement in the tensile and/or compressive zone of the section.

Prestressed concrete is today being used in combination with substantially large amounts of non-prestressed reinforcement. Newer concrete materials such as high strength, lightweight, fiber reinforced or polymer modified concretes, where all the Code assumptions related to ultimate compressive strain and dimensions of the stress block may not apply, are also being used.

Thus, there is an increasing need in these types of applications for a tool to predict flexural capacity of the section and more importantly to predict with enough accuracy, curvatures, rotations and deflections at ultimate. It should be based on a more accurate analysis in which the actual stress-strain properties of the materials involved are taken into consideration.
Such a procedure (referred to as "strain compatibility") is suggested by the ACI specifications (Reference 1, Section 18.7) in order to determine the stress in the prestressing steel ($f_{ps}$) at ultimate behavior; it is accepted in all cases in lieu of the more approximate Code formula for $f_{ps}$ and required when the effective prestress is less than one-half the steel strength and/or when the steel stress-strain curve does not conform with specified ASTM standards.

Furthermore, the use of a more accurate analysis may lead to substantial savings in the amount of prestressing steel required which more than offset the additional cost in design.

The purpose of this paper is to present a simplified procedure to analyze the behavior at ultimate of bonded prestressed and partially prestressed concrete structural elements in which the non-linear behavior of the prestressing steel is fully accounted for. A computer program was written to implement the above procedure.

For given conditions of reinforcement the program leads to the values of stress and strain in the prestressing steel at ultimate, the ultimate moment capacity and the corresponding curvature of the section and other relevant information. Corresponding values obtained by strictly applying the ACI specifications are also computed for direct comparison.

The program allows a quantitative assessment of the influence on ultimate behavior of important parameters such as amount of non-prestressed reinforcement, effective prestress, type of prestressing steel, ultimate compressive strain of the concrete, and stress block dimensional factors.

Numerical results are applied to a rectangular and a box (T) section and a typical design example is developed using the ACI code approach, the strain compatibility procedure, and a combination of prestressed and non-prestressed reinforcement.

**Synopsis**

A non-linear analysis procedure which attempts to predict the behavior at ultimate of prestressed and partially prestressed sections is presented.

It takes into consideration the non-linear characteristics of the steel but assumes that the strains and forces in the concrete at ultimate are well approximated by the ACI specifications.

The numerical techniques and the computer program implementing the proposed procedure are rapidly reviewed.

They help analyze the influence on ultimate behavior of major variables such as type of prestressing steel, amount of non-prestressed reinforcement, ultimate compressive strain and stress block dimensional coefficients of the concrete. Examples of applications and typical results are presented.

It is shown that although the ACI recommended procedure to predict ultimate moment capacity is reasonably accurate and conservative, it may lead to substantial error in predicting strain in the prestressing steel and the corresponding curvature of the section.
Non-Linear Analysis

In the non-linear analysis of reinforced concrete elements, two assumptions are generally considered: first that the strain distribution in the section remains linear up to maximum moment capacity and, second, that the actual stress-strain relations of the steel and concrete are known and used.

In order to determine maximum moment capacity the moment versus curvature curve is plotted and the peak value selected. This can be achieved numerically in the most general case first by selecting a value of concrete extreme fiber compressive strain and then finding by iteration a location of neutral axis which simultaneously satisfies compatibility and equilibrium. The process is repeated for increasing values of the extreme fiber compressive strain. By calculating at every step the internal moment and the curvature, the ultimate (maximum) moment and the corresponding strain and curvature at ultimate can be determined. Practically speaking, a substantial computational effort is needed even when a computer is used.

The above exact method was applied to prestressed concrete sections and was found (Appendix I) to lead to conditions at ultimate not significantly different from a less exact method described below. This later method was selected because first it reduces by an order of magnitude the required computations, and second it accommodates the ACI specifications and is, therefore, of more direct use to designers.

The proposed method takes into account the actual stress-strain curve of the prestressing steel but considers that the ACI assumptions leading to the equivalent rectangular stress block in the concrete are representative of the forces in the concrete at ultimate.

The rationale of this approach stems from the fact that in predicting the stress-strain properties of concrete, one usually encounters much more variabilities than for the steel.

The practically constant elastic modulus of the steel as well as the requirements of ASTM standards on proportional limit, yield strength, ultimate strength and elongation make the predicted stress-strain relation of a given steel quite close to, and representative of the actual relation. This approach will be followed in this paper.

Stress and Strain in Steel at Ultimate for Bonded Members

The design assumptions of the ACI-318-71 concerning the linear strain distribution and the concrete stress block at ultimate are followed and could be visualized by referring to Fig. 1. It is also assumed that the stress-strain relation of the prestressing steel is known either graphically or numerically.

It is further assumed that the strain in the top fiber of the concrete section under effective prestress alone is negligible and that if non-prestressed tensile reinforcement is used it has the same center of gravity as the prestressed reinforcement.

Note that in Fig. 1, and later in Fig. 5, the non-prestressed tensile reinforcement for T sections is shown entirely in the web while, in a particular design, it may be partly associated with the flange.

The determination of the actual stress and strain in the prestressing steel at ultimate requires also the knowledge of a relation between stress and strain as derived from compatibility of strain and equilibrium of the section.

The derivation procedure for rectangular sections can be found in Refs. 3, 12-15. It is generalized here to include auxiliary non-prestressed reinforcement and T-type sections.
Fig. 1. Assumed forces and strains at ultimate capacity of prestressed beam sections.

Stress-strain relation as derived from compatibility and equilibrium

Referring to the strain diagram in the concrete at ultimate capacity (Fig. 1), it can be shown\(^{13,14}\) that the distance from the top fiber to the neutral axis is given by:

\[
c = \left( \frac{\varepsilon_u}{\varepsilon_{ps} + \varepsilon_u - \varepsilon_{ce} - \varepsilon_{cc}} \right) d \quad (1)
\]

where

- \(d\) = distance from top fiber to centroid of tensile force
- \(\varepsilon_u\) = extreme fiber compressive strain of concrete at ultimate (assumed equal to 0.003 in ACI Code)
- \(\varepsilon_{ce}\) = compressive strain in concrete at level of steel under effective prestress
- \(\varepsilon_{ps}\) = tensile strain in prestressing steel at ultimate capacity of beam

\(\varepsilon_{cc}\) = tensile strain in prestressing steel at effective prestress

Note, that for a given beam cross section, Eq. (1) is a relation between \(c\) and \(\varepsilon_{ps}\) as all other terms are known. In general, \(\varepsilon_{cc}\) is small as compared to the other terms and could be neglected.

An additional equation translating the equilibrium of tensile and compressive forces in the section at ultimate capacity (Fig. 1) can be written, but its form depends on whether the section behaves as a rectangular section or as a T section.

If the section behaves at ultimate as a rectangular section the equilibrium
condition leads to the following equation:

\[ A_{ps} f_{ps} + A_s f_{ts} = 0.85 f'_c b \beta c + A'_s f'_{cs} \]  

(2)

where \( f_{ps} \) is the tensile stress in the prestressing steel at ultimate moment capacity, \( f_{ts} \) is the tensile stress in the non-prestressed tensile steel \( (f_{ts} \leq f_y) \) and \( f'_{cs} \) is the compressive stress in the non-prestressed compression steel \( (f'_{cs} \leq f'_y) \).

Generally, \( f_{ts} \) equals \( f_y \) at ultimate while \( f'_{cs} \) can be substantially different. Other symbols of Eq. (2) are standard and can be found in Appendix III.

Note that this equation can easily be modified to accommodate concretes for which the dimensional factors of the stress block are different.

Eliminating \( c \) from Eq. (1) and (2) leads to the following relation between \( f_{ps} \) and \( \epsilon_{ps} \):

\[
0.85 f'_c b \beta c d \frac{0.85 f'_c b \beta c d}{A_{ps}} + \frac{\epsilon_u}{\epsilon_{ps} + \epsilon_u - \epsilon_{sc} - \epsilon_{ce}} + \frac{(A'_s f'_{cs} - A_{ts} f'_{ts})}{A_{ps}}
\]

(3)

In this relation \( f'_{cs} \) and \( f_{ts} \) are functions of \( \epsilon_{ps} \) provided their actual value is less than their yield limit.

Referring to the strain diagrams in Fig. 1 it can be shown that \( f'_{cs} \) and \( f_{ts} \) are given by:

\[
\begin{align*}
(f'_{cs} = E_s \left[ \epsilon_u + \frac{d'}{d} \times (\epsilon_{se} + \epsilon_{ce} - \epsilon_u) - \frac{d'}{d} \epsilon_{ps} \right] \leq f'_y \\
(f_{ts} = E_s \left[ \epsilon_u - \epsilon_{so} - \epsilon_{ce} \right] \leq f_y
\end{align*}
\]

(4)

where \( E_s \) is the modulus of elasticity of conventional non-prestressed reinforcement.

It can be shown that if the section behaves at ultimate as a T section the relation between \( f_{ps} \) and \( \epsilon_{ps} \) takes the following form:

\[
f_{ps} = \left[ \frac{0.85 \beta \epsilon'_{c} d \epsilon_{d} d}{A_{ps}} \right] \times \left[ \frac{\epsilon_u}{\epsilon_{ps} + \epsilon_u - \epsilon_{sc} - \epsilon_{ce}} \right] + \frac{0.85 f'_c (b - b_w) h_t + A'_s f'_{cs} - A_{ts} f_{ts}}{A_{ps}}
\]

(5)

where \( f'_{cs} \) and \( f_{ts} \) can also be replaced by their values from Eq. (4).

Whether a rectangular section or a T section behavior exists at ultimate, the relation between \( f_{ps} \) and \( \epsilon_{ps} \) given by Eq. (3) or (5) can be reduced to the following most general form:

\[
f_{ps} = \frac{A}{\epsilon_{ps} + B} + C \epsilon_{ps} + D
\]

(6)

where \( A, B, C, \) and \( D \) are parameters depending on materials characteristics, effective prestress and geometric properties of the section.

The graphical representations of Eq. (6) corresponding to various cases where \( C \) and \( D \) can take on zero or non-zero values are shown in Fig. 2.

### Steel stress-strain relation

Eq. (6) relating the stress \( f_{ps} \) and the strain \( \epsilon_{ps} \) in the prestressing steel at ultimate capacity of the section has been primarily derived from considerations of equilibrium and linear strain distribution in the concrete section.

Actual values of \( f_{ps} \) and \( \epsilon_{ps} \) must also satisfy the stress-strain relation of the prestressing steel. The problem is the same as having two equations with two unknowns and solving for these unknowns.

As these equations are generally of a high order, charts are developed for...
their solution such as, for example, in References 15 and 17 and in the PCI Design Handbook (Reference 18, Fig. 5.25); they lead, for purely prestressed beams, to the stress in the prestressing steel at ultimate; but they do not cover the case where auxiliary non-prestressed reinforcement is used.

In this case a graphical solution (Fig. 2) can, in general, be obtained by determining the point of intersection of a curve representing Eq. (6) with the stress-strain curve of the prestressing steel.

The graphical solution, however, is not very suitable if a repetitive analysis is to be performed with many cross-sectional shapes, types of prestressing steel and different values of prestressed and non-prestressed reinforcement ratios. The use of a computerized numerical technique such as the one explained below becomes a necessary tool for solution.

### Fig. 2. Typical relations between stress and strain in prestressing steel as derived from equilibrium and compatibility.

<table>
<thead>
<tr>
<th>Typical Relation as Derived from Equilibrium</th>
<th>Graphical Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ f_{ps} = \frac{A}{\varepsilon_{ps} + B} ]</td>
<td><img src="" alt="Equilibrium Graph" /></td>
</tr>
<tr>
<td>( A &gt; 0 )</td>
<td><img src="" alt="Stress Strain Curve" /></td>
</tr>
<tr>
<td>( B &lt; 0 )</td>
<td></td>
</tr>
<tr>
<td>Rectangular beams with prestressed reinforcement only, ( A_{ps} ).</td>
<td></td>
</tr>
<tr>
<td>[ f_{ps} = \frac{A}{\varepsilon_{ps} + B} + C ]</td>
<td><img src="" alt="Equilibrium Graph" /></td>
</tr>
<tr>
<td>( C \geq 0 )</td>
<td><img src="" alt="Stress Strain Curve" /></td>
</tr>
<tr>
<td>Rectangular beam with ( A_{ps}, A_s ) and/or ( A_s' ) when ( f_{ts} = f_y ) and/or ( f'_{cs} = f'_y ).</td>
<td></td>
</tr>
<tr>
<td>TEE beam: with ( A_{ps} ) only</td>
<td></td>
</tr>
<tr>
<td>TEE beam: with ( A_{ps}, A_s ) and/or ( A_s' ) when ( f_{ts} = f_y ) and/or ( f'_{cs} = f'_y ).</td>
<td></td>
</tr>
<tr>
<td>[ f_{ps} = \frac{A}{\varepsilon_{ps} + B} + C \varepsilon_{ps} + D ]</td>
<td><img src="" alt="Equilibrium Graph" /></td>
</tr>
<tr>
<td>Rectangular beam with ( A_{ps}, A_s ) and/or ( A_s' ) when ( f_{ts} &lt; f_y ) and/or ( f'_{cs} &lt; f'_y ).</td>
<td></td>
</tr>
<tr>
<td>TEE beam with ( A_{ps}, A_s ) and/or ( A_s' ) when ( f_{ts} &lt; f_y ) and/or ( f'_{cs} &lt; f'_y ).</td>
<td></td>
</tr>
</tbody>
</table>
Numerical Technique of Solution

There are two aspects of the numerical technique as used here; the first is to find a numerical equation which adequately represents the stress-strain curve of the prestressing steel especially in its non-linear portion; the second is to find the point of intersection of two curves, namely, Eq. (6) and the stress-strain relation.

This point of intersection leads to the solution sought, i.e., the compatible values of $f_{ps}$ and $\epsilon_{ps}$ at ultimate.

Curve fitting

The stress-strain curve of the prestressing steel was approximated by three equations, two of them linear, representing the initial and final portion of the curve, and one non-linear representing the middle portion (see Fig. 3).

Using a polynomial fit for the middle portion, the stress-strain curve of a typical prestressing strand (diameter 0.50 in.; strength = 270 ksi; elastic modulus = 27,890 ksi; ultimate strain = 0.070; produced by Kurt Orban Company) was represented by the following equations:

$$f = \begin{cases} 27890\epsilon & \text{for } 0 \leq \epsilon \leq 0.006 \\ 259 + 196.43(\epsilon - 0.014) & \text{for } 0.014 < \epsilon \leq 0.070 \\ 167.34 + 254.68443[100(\epsilon - 0.006)] - 37.19658[100(\epsilon - 0.006)]^2 - 419.53833[100(\epsilon - 0.006)]^3 + 308.89136[100(\epsilon - 0.006)]^4 & \text{for } 0.006 < \epsilon \leq 0.014 \end{cases}$$

The maximum error between actual stress value and the value estimated from the polynomial equation is around 0.4 percent.

Fig. 3. Stress-strain curves of prestressed and conventional steel as used in numerical fits.
Another type of prestressing steel, namely, a prestressing bar (diameter = 1.25 in.; strength = 160 ksi; elastic modulus = 28,750 ksi; ultimate strain ≈ 0.055, produced by Stressteel Corp.) was also used in the numerical examples.

Its actual stress-strain curve was approximated by the following linear and polynomial equations:

\[
f = 28750\varepsilon \text{ for } 0 \leq \varepsilon \leq 0.004
\]
\[
f = 144.90 + 503.33(\varepsilon - 0.010) \text{ for } 0.010 < \varepsilon \leq 0.040
\]
\[
f = 115.0 + 244.12300\left[\frac{100(\varepsilon - 0.004)}{100(\varepsilon - 0.004)}\right] - 896.22266
\]
\[
\left[100(\varepsilon - 0.004)\right]^2 + 1521.94959\left[100(\varepsilon - 0.004)\right]^3 - 946.56592
\]
\[
\left[100(\varepsilon - 0.004)\right]^4 \text{ for } 0.004 < \varepsilon \leq 0.010
\]

![Diagram](image)

Fig. 4. Flow chart to determine stress in prestressing steel at ultimate for bonded members.
Note: In the exact analysis $f_y$ and $f'_y$ should be replaced by $f_{ts} < f_y$ and $f'_{cs} < f'_y$. 
Here the maximum error between input stress and stress estimated from the polynomial is around 1 percent.

**Intersection of two curves**

The point of intersection of the two curves representing Eq. (6) and (7) or (8) was obtained using the Newton-Raphson's numerical technique.$^{19,20}$

In this technique convergence is very fast (quadratic), and only a few iterations are needed. For the results described below, the stress in the pre-stressing steel at ultimate was determined with an accuracy of 0.1 ksi.

**Test Examples**

A computer program was written implementing the preceding analysis. For a given prestressed and partially pre-stressed rectangular or T section, the program leads to the values of stress and strain in the pre-stressing steel at ultimate behavior as well as the ultimate moment, curvature and other relevant parameters. Corresponding values using the design procedure suggested by the ACI Code are also computed for direct comparison.

Two relevant flow charts of the computer program may be of interest to the reader and are shown in Figs. 4 and 5.

The first chart describes the procedure to determine the stress in the pre-stressing steel at ultimate capacity by strain compatibility and the second one describes the steps for the ultimate moment design of prestressed and partially prestressed sections according to the ACI specifications.$^1$ This latter chart can also be used to determine the ultimate moment capacity according to the procedure described in this paper, provided the value of $f_{ps}$ and the corresponding values of $f_t = f_y$ and $f_{ca} = f_y$ as derived from strain compatibility are used.

Two beam cross-sectional shapes, a rectangular and box-type section (which behave as a T at ultimate) are used to illustrate the results obtained by the above non-linear analysis. Their dimensions are given in Fig. 6.

In the following figures, where observed results are described, reference to Rectangular or T relates directly to one of these two beams and indicate trends expected on similar shapes of beams.

Furthermore, reference to $f_{ps} = 270$ ksi and $f_{ps} = 160$ ksi relates to the pre-stressing strand and the pre-stressing bar described in the previous section on “curve fitting,” for which the stress-strain relationships are respectively given by Eq. (7) and (8).

Conventional non-prestressed reinforcement is assumed to have a yield strength of 60 ksi and to have an elasto-plastic stress-strain relation as shown in Fig. 3. A typical design example is treated in Appendix II.

---

**Fig. 6. Cross sections of beams used in examples.**

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Analysis of Results

Most relevant test results are described graphically in Figs. 7 to 12. The first series of results relates to the variation of the stress in the prestressing steel at ultimate behavior of the section, $f_{ps}$.

Fig. 7 shows the variation of $f_{ps}$ versus the prestressed reinforcement ratio for two different values of concrete compressive strength $f'c$ and two values of effective prestress, $f_{se}$.

Also, the ACI recommended value of $f_{ps}$ for bonded members is plotted on the same figure and corresponds to the following equation:

$$f_{ps} = f_{pu} \left[ 1 - 0.5 \rho_p \frac{f_{pu}}{f'c} \right] \quad (9)$$

It can be seen that the ACI equation, as usually expected from a code specification is quite conservative. For example, for $\rho_p = 0.4$ percent a difference of 7 percent in stress is observed between the value of $f_{ps}$ proposed by ACI and the value predicted by the present analysis.

However, this relatively small difference in stress leads to more than 35 percent difference in strain and a similar large discrepancy may be expected in computing curvature, rotation or deflection.

Note, too, that a 7 percent difference in stress leads to about an equal difference in the amount of steel required and the corresponding cost (see also the example in Appendix II).

The influence on $f_{ps}$ of a concrete
having a compressive strain at ultimate
different from the ACI recommended
value of 0.003 is explored in Fig. 8.

It can be seen that the code recom-
mended value of $f_{ps}$ allows for a rea-
sonable range of variation of $\epsilon_u$ with-
out loss of safety. However, here too a
slight difference in the value of $f_{ps}$ may
lead to a substantial difference in strain
and curvature.

Fig. 9 shows the influence of non-
prestressed tensile and compressive re-
forcement on the value of $f_{ps}$ for the
rectangular and the T section of Fig. 6.
The ACI recommended value of Eq.
(9) is also plotted for comparison.

It can be observed that the trend
predicted by the ACI equation for the
rectangular section is well followed
over a wide range of variation of the
prestressed reinforcement ratio $\rho_p$. This
is not the case, however, for the T
section at relatively large and not very
common values of $\rho_p$.

This result may be due to the defi-
nition of the prestressed reinforcement
t ratio for T sections, $\rho_p = \frac{A_{ps}}{bd}$,
where $b$ is the flange width. This ratio
is the same as for an equivalent rec-
tangular section of width $b$.

Actually if $\rho_p$ keeps increasing and
the section behaves as a T section, the
corresponding reinforcement ratio of
the web alone $\frac{A_{ps}}{b_w b}$ increases at a
much faster rate than $\rho_p$ and thus the
stress and strain in the prestressing
steel decrease at a faster rate than for
the equivalent rectangular section.

Similar trends on T sections were
observed when different prestressing

![Fig. 8. Influence of ultimate concrete compressive strain on $f_{ps}$.](image-url)
steels or different effective prestress were used, but they are not shown herein. It is interesting to note that this is a unique case where the ACI Code is on the unsafe side.

The influence of prestressed and non-prestressed reinforcement ratios on the curvature of rectangular and T sections at ultimate is shown in Fig. 10. The curvature has been computed from the following equation:

$$\Phi = \frac{c}{d} = \frac{\epsilon_{ps} + \epsilon_{cu} - \epsilon_{ce}}{d}$$  \hspace{1cm} (10)

which could also be derived from Eq. (1).

If on Fig. 10 one plots the curvature at ultimate using $\epsilon_{ps}$ corresponding to the $f_{ps}$ predicted by ACI Eq. (9) it will almost follow the lower curve for both the rectangular and T section. However, it was not plotted to keep the figure clear.
It can be concluded from Fig. 10 that the higher the prestressed reinforcement ratio the smaller the influence of non-prestressed reinforcement on curvature and the more accurate the curvature predicted from ACI Eq. (9).

In the numerical tests carried in this study with various beam cross-sectional shapes and ratios of prestressed and non-prestressed steel, it was generally observed that the theoretical ultimate moment capacity predicted by ACI was less than 10 percent smaller than the value derived from the present analysis.

This conclusion confirms the 7 percent average difference observed in actual tests as reported by Mattock. However, as can be observed in Fig. 11, a severe restriction on ultimate moment is applied by ACI for “over-reinforced” beams with prestressed and
non-prestressed tensile reinforcement. In this figure a capacity modification factor of 0.9 has been used for all values of moments.

In computing the reinforcement index according to ACI, it is assumed that the compressive steel yields; this assumption does not seriously affect the ultimate moment capacity. Fig. 12 illustrates the actual variation of the stress in the non-prestressed compressive steel at ultimate for the rectangular section.

It can be seen that the higher the amount of compressive reinforcement and the lower the strength of the prestressing steel, the higher the prestressed reinforcement ratio necessary to assure yielding of compressive steel.

In order to illustrate the influence of non-prestressed tensile or compressive reinforcement we have used each of them separately in combination with prestressed reinforcement. Let us note that if they are used simultaneously they tend to compensate each other’s effects on the stress in the prestressing steel and the curvature at ultimate.

Fig. 11. Comparison of ultimate moment by ACI Code and present analysis.
Fig. 12. Typical variation of the stress in non-prestressed compressive steel.

**Conclusion**

The use of computerized numerical techniques to model and analyze the behavior of prestressed and partially prestressed concrete beams in the non-linear range may become a useful and efficient design tool when such information is necessary.

Once a model has been proven satisfactory, the influence of many parameters can be qualitatively and quantitatively analyzed, thus substantially reducing the amount of experimental work which would otherwise be required.

From a practical viewpoint, the cost of a more accurate method of analysis may be largely offset by the savings on the amount of steel used.

**NOTE**

Three Appendices follow:
Appendix I presents a numerical comparison of four methods of analysis. Appendix II contains a design example solved using three different approaches. Appendix III summarizes the notation used. Lastly, a list of references is given.

Discussion of this paper is invited. Please forward your discussion to PCI Headquarters by July 1, 1977.
APPENDIX I—COMPARISON OF METHODS OF ANALYSIS

The exact method of non-linear analysis was applied to prestressed concrete beams using the stress-strain curve of prestressing steel say from Eq. (7), and that of concrete provided by Popovics\(^9,10\) and given by:

\[
\left(\frac{f}{f_o}\right) = \left(\frac{\varepsilon}{\varepsilon_o}\right) \frac{n}{n-1} + \left(\frac{\varepsilon}{\varepsilon_o}\right)^n
\]  

(11)

where \(f_o\) is the peak stress in the concrete (i.e., \(f\_c\)), \(\varepsilon_o\) the corresponding strain, and \(n\) a material parameter.

The recommended values of \(\varepsilon_o\) and \(n\) for normal weight concrete\(^10\) are given by:

\[
\varepsilon_o = 2.7 \times 10^{-4} \times \sqrt[4]{f_o}
\]  

\[
n = 0.4 \times 10^{-2} \times f_o + 1
\]  

and were used in the example described in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Typical results using different methods of analysis.*</th>
</tr>
</thead>
<tbody>
<tr>
<td>REINFORCEMENT RATIOS</td>
</tr>
<tr>
<td>-----------------------</td>
</tr>
</tbody>
</table>
| $\rho_p = 0.002$  
$\rho = 0$  
$\rho_p' = 0$ | Exact Nonlinear  
Present Nonlinear  
PCI Handbook  
ACI** | 2348  
2332  
2286 | 261.1  
260.9  
255.4 | 0.02472  
0.02205  
0.01172 | 10.6  
9.31  
4.40 |
| $\rho_p = 0.002$  
$\rho = 0.002$  
$\rho_p' = 0$ | Exact Nonlinear  
Present Nonlinear  
ACI | 2859  
2818  
2781 | 260.2  
260.0  
255.4 | 0.02014  
0.01843  
0.01172 | 8.40  
7.59  
4.40 |
| $\rho_p = 0.002$  
$\rho = 0$  
$\rho_p' = 0.002$ | Exact Nonlinear  
Present Nonlinear  
ACI | 2357  
2359  
2320 | 261.5  
261.2  
255.4 | 0.02676  
0.02425  
0.01172 | 11.6  
10.36  
4.40 |
| $\rho_p = 0.004$  
$\rho = 0$  
$\rho_p' = 0$ | Exact Nonlinear  
Present Nonlinear  
PCI Handbook  
ACI | 4374  
4310  
4067 | 258.5  
257.5  
240.8 | 0.01370  
0.01255  
0.00968 | 5.30  
4.71  
3.35 |
| $\rho_p = 0.004$  
$\rho = 0.004$  
$\rho_p' = 0$ | Exact Nonlinear  
Present Nonlinear  
ACI | 5188  
5060  
4917 | 256.1  
251.3  
240.8 | 0.01200  
0.01084  
0.00968 | 4.50  
3.90  
3.35 |
| $\rho_p = 0.004$  
$\rho = 0$  
$\rho_p' = 0.004$ | Exact Nonlinear  
Present Nonlinear  
ACI | 4501  
4464  
4197 | 260.0  
259.6  
240.8 | 0.01917  
0.01708  
0.00968 | 8.28  
6.87  
3.35 |

*For the rectangular section of Fig. 4 with $f_c = 5000$ psi; $f_{pu} = 270$ ksi; $f_{sc} = 0.55 f_{pu}$ and $f_y = f_{\gamma} = 60$ ksi.

**For this method the strain was obtained from the corresponding stress on the stress-strain curve. The stress was computed from Equation (9).
In the less than exact non-linear analysis described in this study it was assumed that the forces and strains in the concrete at ultimate are well approximated by the ACI assumptions (Fig. 1) while the exact stress-strain curve of the prestressing steel [see Eq. (7)] is considered.

Using the ACI Code allows the approximation of not only the forces in the concrete at ultimate but also in the prestressing steel through the use of Eq. (9).

Typical results obtained by using the three above described methods are shown in Table 1. Also shown are the stresses in the prestressing steel for purely prestressed sections as derived from the PCI Design Handbook, Fig. 5.2.5.

It can be seen that although comparable values of ultimate moments are obtained by the three procedures, substantial differences in strains and curvatures at ultimate exist between the ACI method and the others.

It can also be observed that the exact non-linear analysis and the less than exact present non-linear analysis lead to substantially similar results over a wide range of variation of prestressed and non-prestressed reinforcement ratios.

Note that the results of the exact non-linear analysis are dependent on the equation predicting the stress-strain curve of the concrete. Eq. (11), for example, may not be accurate enough in predicting the descending branch of the stress-strain curve in the range of interest.

**APPENDIX II—DESIGN EXAMPLE**

Consider a prestressed concrete rectangular slab with dimensions given in Fig. 6 (assuming a 1-ft wide strip) for which: $f_c' = 5000$ psi; $f_{pu} = 270$ ksi; $f_{y} = 0.55 f_{pu}$ and $f_y = f_y = 60$ ksi.

Assume that the ultimate moment required at the critical section is equal to 425 ft-kips and that conditions at ultimate control the design. Several solutions can be derived as follows:

(a) Using the ACI Code approach (see flowchart Fig. 5) for purely prestressed sections and after few iterations leads to:

- $\rho_p = 0.0055$
- $\epsilon_p = 0.0890$
- $A_{ps} = 1.38$ sq in.
- $q = 0.253$
- $f_{ps} = 229.90$

Factorized moment capacity = $\phi M_u = 427$ ft-kips (ok).

(b) Using the strain compatibility procedure described in the text leads to:

- $\rho_p = 0.005$
- $\epsilon_{ps} = 0.01085$
- $A_{ps} = 1.26$ sq in.
- $q = 0.251$
- $f_{ps} = 251.3$ ksi

while the more accurate analysis with the same prestressed reinforcement ratio leads to $f_{ps} = 250.3$ ksi and an area of non-prestressed tensile steel

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\( A_s = 1.09 \text{ sq in. about 12 percent smaller than in the previous case.} \)

Note that the PCI Design Handbook does not include a chart for \( f_{ps} \) when auxiliary non-prestressed reinforcement is used. Note also that most of the above results can be estimated from Figs. 9 and 11.

### APPENDIX III—NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = \beta_1 c )</td>
<td>depth of equivalent rectangular stress block</td>
</tr>
<tr>
<td>( A_{ps} )</td>
<td>area of prestressing steel</td>
</tr>
<tr>
<td>( A_s )</td>
<td>area of non-prestressed tension reinforcement</td>
</tr>
<tr>
<td>( A'_s )</td>
<td>area of non-prestressed compression reinforcement</td>
</tr>
<tr>
<td>( b )</td>
<td>width of flange of flanged member or of rectangular member</td>
</tr>
<tr>
<td>( b_w )</td>
<td>width of web of flanged member</td>
</tr>
<tr>
<td>( c )</td>
<td>distance from extreme compression fiber to neutral axis</td>
</tr>
<tr>
<td>( d )</td>
<td>distance from extreme compression fiber to centroid of prestressing steel or to combined centroid when non-prestressed tension reinforcement is included</td>
</tr>
<tr>
<td>( d' )</td>
<td>distance from extreme compression fiber to centroid of compressive reinforcement</td>
</tr>
<tr>
<td>( E_s )</td>
<td>modulus of elasticity of non-prestressed steel if different from ( E_{ps} )</td>
</tr>
<tr>
<td>( E_c )</td>
<td>modulus of elasticity of concrete</td>
</tr>
<tr>
<td>( E_{ps} )</td>
<td>modulus of elasticity of prestressing steel</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>strain in general</td>
</tr>
<tr>
<td>( \varepsilon_{ce} )</td>
<td>concrete strain at centroid of prestressing steel due to effective prestress</td>
</tr>
<tr>
<td>( \varepsilon_{cu} )</td>
<td>equivalent concrete strain at centroid of prestressing steel, at ultimate flexural capacity</td>
</tr>
<tr>
<td>( \varepsilon_{ps} )</td>
<td>strain in prestressing steel at ultimate flexural capacity</td>
</tr>
<tr>
<td>( \varepsilon_{se} )</td>
<td>strain in prestressing steel due to effective prestress ( f_{se} )</td>
</tr>
<tr>
<td>( \varepsilon_u )</td>
<td>concrete strain at ultimate on extreme compression fiber</td>
</tr>
<tr>
<td>( f )</td>
<td>stress in general</td>
</tr>
<tr>
<td>( f_{ps} )</td>
<td>average stress in prestressing steel at ultimate load capacity, psi</td>
</tr>
<tr>
<td>( f_{pu} )</td>
<td>ultimate strength of prestressing steel, psi</td>
</tr>
<tr>
<td>( f_{se} )</td>
<td>effective stress in prestressing steel after losses, psi</td>
</tr>
<tr>
<td>( f_{ts} )</td>
<td>tensile stress in the non-prestressed tensile reinforcement, ksi, ( f_{ts} = f_y )</td>
</tr>
<tr>
<td>( f_y )</td>
<td>yield strength of non-prestressed conventional reinforcement in tension</td>
</tr>
<tr>
<td>( f_c )</td>
<td>specified compressive strength of concrete, psi</td>
</tr>
<tr>
<td>( f_{cs} )</td>
<td>compressive stress in non-prestressed compression reinforcement, ( f_{cs} \leq f_y )</td>
</tr>
<tr>
<td>( f'_y )</td>
<td>yield strength of non-prestressed conventional reinforcement in compression</td>
</tr>
<tr>
<td>( \phi )</td>
<td>capacity modification factor</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>curvature of section</td>
</tr>
<tr>
<td>( h_f )</td>
<td>average thickness of flange of a flanged member</td>
</tr>
<tr>
<td>( q = \omega + \omega' )</td>
<td>reinforcing index of a rectangular member or of a flanged member designed as a rectangular member</td>
</tr>
<tr>
<td>( \rho = A_s/bd )</td>
<td>ratio of non-prestressed tension reinforcement</td>
</tr>
<tr>
<td>( \rho' = A'_s/bd )</td>
<td>ratio of compression reinforcement</td>
</tr>
<tr>
<td>( \rho_p = A_{ps}/bd )</td>
<td>ratio of prestressed reinforcement</td>
</tr>
<tr>
<td>( \omega = \rho f_{ps}/f_c )</td>
<td></td>
</tr>
<tr>
<td>( \omega' = \rho' f_{ps}/f_c )</td>
<td></td>
</tr>
<tr>
<td>( \omega_p = \rho_p f_{ps}/f'_c )</td>
<td></td>
</tr>
</tbody>
</table>
References

1. ACI Committee 318, "Building Code Requirements for Reinforced Concrete (ACI 318-71)," American Concrete Institute, Detroit, 1971, Chapter 18.


11. ACI Committee 318, "Commentary on Building Code Requirement for Reinforced Concrete (ACI 318-71)," American Concrete Institute, Detroit, 1971, Chapter 18.


18. PCI Design Handbook—Precast and Prestressed Concrete, Prestressed Concrete Institute, Chicago, 1971, Fig. 5.2.5.
