Fatigue Properties of Prestressing Strand

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SYNOPSIS

The fatigue properties of 7/16-in. diameter seven-wire prestressing strand were studied in an experimental investigation involving static tests, constant cycle fatigue tests, and cumulative damage tests on approximately 150 specimens. Equations were derived for the probable fatigue life of strand elements under repeated loading of either constant or varied magnitude.

The values of the test variables were chosen so that the range of applicability of the resulting equations would cover the most important practical situations arising in the study of the fatigue life of prestressed concrete flexural members.

INTRODUCTION

The fatigue properties of under reinforced prestressed concrete beams are governed primarily by the fatigue properties of the reinforcing steel. Before an estimate can be made of the fatigue life of such a member, it is therefore necessary to have detailed information on the fatigue properties of the tension reinforcement. This paper presents the results of an experimental study of the fatigue properties of 7/16-in. diameter seven-wire strand, a type of prestressing steel used extensively in the United States in the manufacture of pretensioned prestressed concrete members.

The strand fatigue tests were divided into two groups. The constant cycle tests comprising the first group were designed to provide an empirical relation between minimum and maximum stress level and probable fatigue life. In the cumulative damage tests comprising the second group, the specimen was subjected to a fatigue loading which fluctuated between a constant minimum stress level and either two or three different maximum stress levels. These tests provided data on the fatigue life of strand elements subjected to varied patterns of repeated loading. Static tests were also conducted to determine the stress-strain properties and static strength of the strand.

In the description of the strand fatigue tests and in the analysis of results which follow, maximum and minimum stress levels and load levels are stated for convenience as percentages of the static ultimate strength.

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NOTATION

D	standard deviation of log N
N, n	number of cycles
$\overline{N}$	mean fatigue life
$\overline{\log N}$	mean of log N
P	probability of strand failure at or before N cycles
R	stress interval: $R = S_{max} - S_L$
S_L	fatigue limit corresponding to S_{min}
S_{max}, S_{01}, S_{02}	maximum stress levels in a repeated load cycle
S_{min}	minimum stress level in a repeated load cycle
S_{pred}	predominant stress level in a cumulative damage test
X	log N
Z	$\dfrac{\log N - \overline{\log N}}{D}$
α, β, γ	parameters to define the shape of the load blocks for cumulative damage tests

TEST VARIABLES

The constant cycle fatigue tests were conducted with minimum stress levels of 40 and 60 percent of the static ultimate strength. Various maximum stress levels were chosen to give fatigue lives varying between 50,000 and 5 million cycles for each minimum stress level. Apart from several tests which yielded fatigue lives outside of this main region of interest, at least six replications of each test were made. Details are given in Table 1 of the different values used for maximum and minimum stress levels and of the number of test replications. One test, with minimum and maximum stress levels of 60 and 80 percent, respectively, was replicated 20 times in order to obtain information not only on mean fatigue life but also on the manner in which the different values of fatigue life were distributed around the mean.

The cumulative damage tests were conducted by repeatedly applying a block of load cycles to the specimen until it failed in fatigue. A constant minimum stress level of either 40 or 60 percent was maintained in each block of load cycles, while the maximum stress varied between two or three different values, as shown in Fig. 1. The range of variables used in the tests are given in Table 1.

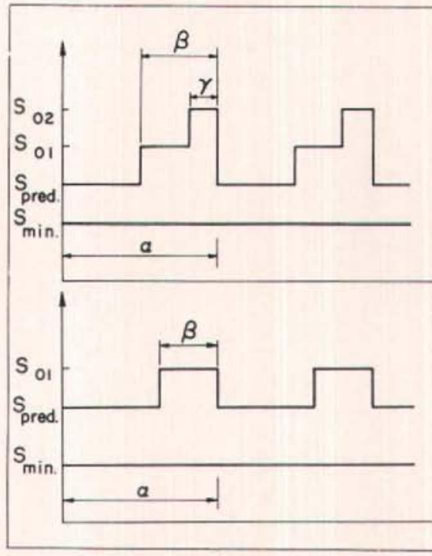

Fig. 1—Load Blocks for Cumulative Damage Tests on Strands

The smallest load cycles in any block are also the most frequently occurring and are referred to as the *design* or *predominant* loading. The larger, less frequently occurring load cycles are regarded as overloadings. In some tests the predominant load was smaller than the fatigue limit indicated by the constant cycle test data, in others it was higher. The over-

Table 1—Strand Fatigue Test Variables

(a) Constant Cycle Tests

Group	Stress Levels % Static Ult.		No. Replications
	S_{min}	S_{max}	
A	40	70	6
B	40	65	6
C	40	60	6
D	40	57.5	6
E	60	85	6
F	60	80	20
G	60	75	7

(b) Cumulative Damage Tests; Two Maximum Stress Levels

Test No.	S_{min}, % Static Ult.	Max. Stress Levels, % Static Ult.		Size & Shape of Stress Block*		No. Replications
		S_{pred}	S_{01}	α	β/α	
4AA	40	40	70	30,000	0.40	2
4BA	40	60	70	30,000	0.40	2
4BB	40	60	70	150,000	0.40	2
4BC	40	60	70	10,000	0.40	2
3AA	60	65	85	30,000	0.25	2
3BA	60	70	85	30,000	0.25	3
3CA	60	75	85	22,500	0.25	2
3DA	60	65	85	22,500	0.40	2
3EA	60	75	85	15,000	0.40	2
3FA	60	80	85	15,000	0.25	10
3AB	60	65	85	300,000	0.25	4
3AC	60	65	85	10,000	0.25	2

(c) Cumulative Damage Tests; Three Maximum Stress Levels

Test No.	S_{min}, % Static Ult.	Max. Stress Levels, % Static Ult.			Size & Shape of Stress Block*			No. Replications
		S_{pred}	S_{01}	S_{02}	α	β/α	γ/β	
5AA	60	60	80	85	60,000	0.40	0.40	2
5BA	60	75	80	85	30,000	0.40	0.40	3
5CA	60	75	80	85	30,000	0.25	0.40	4
6AA	40	60	65	70	20,000	0.40	0.40	2
6BA	40	60	65	70	30,000	0.40	0.25	3
6CA	40	50	60	70	30,000	0.40	0.40	2

* for meaning of α, β, γ, see Fig. 1.

loadings, however, were always larger than the fatigue limit.

In one series of tests the main variable was the number of cycles contained in the load block. Otherwise, the size of the load blocks was chosen to be approximately one-tenth of the expected fatigue life.

The cumulative damage tests were replicated either two or three times, but in one case ten replications were made to observe the distribution of the values about the mean.

SPECIMENS

The strand test specimens were taken from a 1500-ft. length of 7/16-in. diameter strand. The strand was cut into 74 pieces approximately 20 ft. long; two specimens were taken from each length and were numbered consecutively in order of use. Thus, each specimen has a length number, prefixed by the letter *L*, and a test number, prefixed by the letter *S*; for example, *L*36-*S*45, etc. To minimize the effect of possible variations in material properties along the 1500-ft. sample, the test lengths were used in random sequence.

The specimens were held with a device which was designed to minimize stress concentrations and hence prevent premature fatigue failure in the gripping region. After a number of different methods had been tried, a gripping arrangement was finally adopted in which the force in the test piece was transmitted partly through a cement-grout bond anchorage and partly through a Strandvise anchorage at the end of the specimen. Details of the grip are shown in Fig. 2.

The specimens were prepared in pairs. A 20-ft. length of strand was tensioned to 70 percent of the static strength in a small prestressing frame, and the elements of the gripping devices were assembled around it. When the force in the strand was released, the Strandvises at the end of each specimen retained a force in the test piece of approximately 45

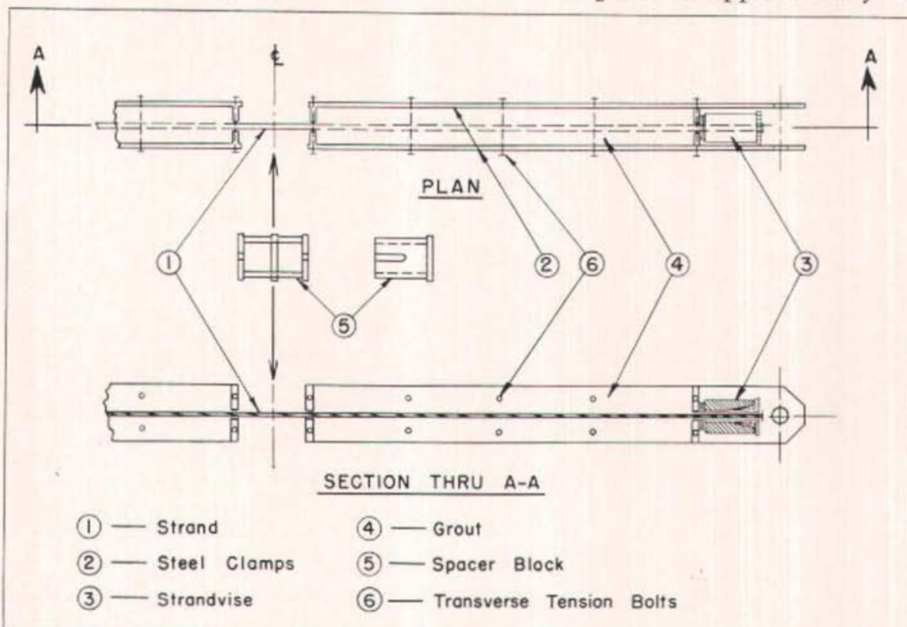

Fig. 2—Strand Gripping Device

percent of the static strength. A stiff sand-cement-water grout, of proportions 1.3:1.0:0.3, was then packed by hand around the strand and the transverse tension bolts. The grout was permitted to cure, and, just prior to testing, the transverse bolts were tightened. The spacer block was removed only when load was applied to the specimen at the beginning of the test.

TEST PROCEDURE

A general view of the strand fatigue testing arrangement is shown in Fig. 3. The specimen was tested in a vertical position, with the lower end pinned to a solid base and the upper end pinned to a horizontal beam. The beam was pinned to a supporting frame at one end and rested at the other on a 22-kip capacity Amsler jack. Dynamic load applied to the jack by an Amsler pulsator induced a dynamic reactive force in the test specimen. The loading was applied at a rate of 500 cycles per minute.

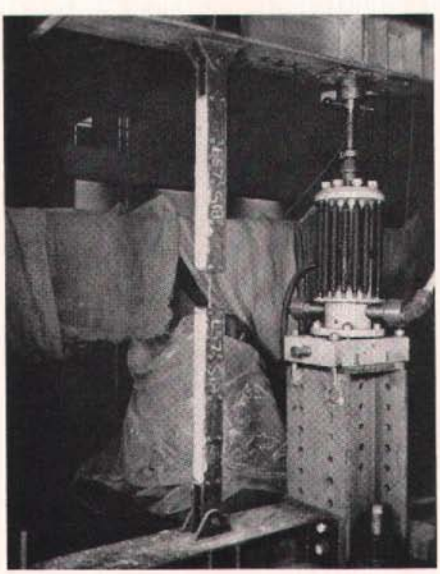

Fig. 3—Strand Fatigue Test Set-Up

In several tests dynamic strain measurements were made with SR-4 gages attached to the upper and lower surfaces of the beam and to individual wires in the specimen. A comparison of dynamic strains with strains measured under static loading indicated that inertial effects were negligible. The test set-up was calibrated so that the jack loads, indicated on dial gages attached to the pulsator, could be used as a measure of the specimen loads.

In the first fatigue tests, which were conducted with 60 percent minimum stress levels, the specimens were positioned halfway between the beam supports. In order to improve the accuracy with which the loads in the specimen were measured, the test set-up was modified to allow specimens to be positioned at the quarter-point closer to the jack. To maintain uniformity in the test results, however, the remaining 60 percent minimum stress level tests were conducted at the half-point position, while all of the 40 percent minimum stress level tests were conducted at the quarter-point.

The static specimens were tested in a 300-kip capacity Baldwin Universal testing machine. The gripping arrangement developed for the fatigue tests was used also for all static strength tests. In tests to determine load-strain curves for the material,

TABLE 2—Static Tests

Specimen	P_{ult}, lbs
L 1 - S 1	28,620
L 2 - S 4	28,675
L 3 - S 5	28,650
L41 - S15	28,450
L57 - S20	28,500
L13 - S51	28,600
L70 - S52	28,520
L70 - S53	28,450

Mean P_{ult} = 28,560 lb.
Standard Deviation = 89 lb.

elongation measurements were made over a 50-in. gage length with Ames dial gages. To compare the average strains measured in the strand with actual steel strains, several tests were conducted with strain gages attached to individual wires in the test specimen.

TEST RESULTS

Static Tests

The results of the static ultimate strength tests are contained in Table 2. All specimens failed in the open length of strand. A mean load-strain relation, obtained from elongation measurements on a 50-in. gage length, is shown in Fig. 4, where it is compared with a load-strain curve obtained using SR-4 gages attached to individual wires. The modulus of elasticity of the strand was found to be 28.0×10^6 psi as against 30.0×10^6 psi for an individual wire.

Constant Cycle Fatigue Tests

The constant cycle fatigue test results are contained in Table 3, where values of the minimum and maximum stress levels are given, in percentages of static strength, together with the number of load cycles at which the first wire in the strand fractured. The results are summarized, for purposes of analysis, in Table 4.

One of the six outside wires was always the first to fail in fatigue. Successive failures occurred in other outside wires until the remaining wires were so overstressed that they failed statically. Those wires which had failed in fatigue could be clearly distinguished by a typical fracture surface containing a crescent shaped fatigue crack.

The number of load cycles separating the first and second wire failures was variable. Sometimes the first and second wires failed almost simultaneously, with complete strand failure following quickly. On other occasions, usually in tests with smaller load range, the interval was large. However, considerable elongation always occurred in the specimen when the first wire failed.

In the majority of the specimens the failure section was in the open region between the gripping pieces. Whenever the failure was within the

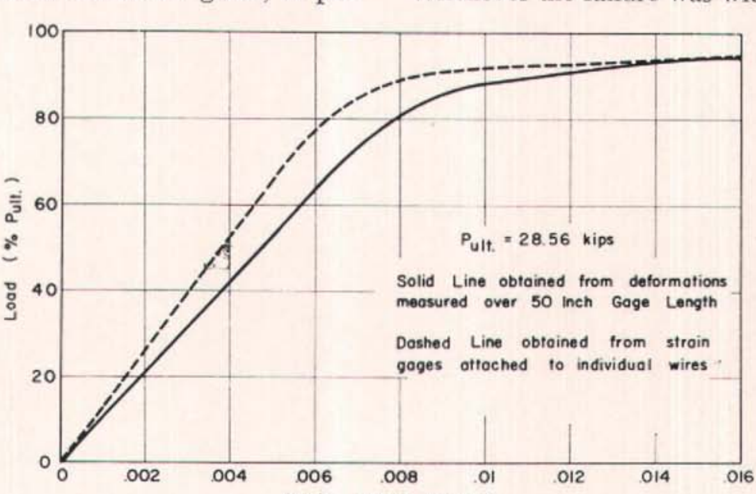

Fig. 4—Load Versus Strain, 7/16 In. Dia. Strand

Table 3—Constant Cycle Strand Test Results

Specimen	S_{min}	S_{max}	N	log N
L 1 - S 2	60	70	460,200[1]	......
L 2 - S 3	60	80	234,400	5.36996
L 4 - S 6	60	80	33,000[2]	......
L 5 - S 8	60	75	425,500	5.62890
L 5 - S 9	60	70	3,306,000	6.51930
L 9 - S10	60	75	304,800	5.48401
L17 - S11	60	70	5,440,600[3]	......
L17 - S12	60	85	103,000	5.01284
L15 - S13	60	80	211,000	5.32428
L15 - S14	60	85	70,000	4.84510
L41 - S16	60	85	88,300	4.94463
L50 - S17	60	75	777,000	5.89042
L50 - S18	60	80	160,000	5.20412
L57 - S19	60	80	170,600	5.23198
L38 - S21	60	80	121,000	5.08279
L38 - S22	60	75	863,000	5.93601
L65 - S25	60	80	159,000	5.20140
L65 - S26	60	85	73,000	4.86332
L36 - S28	60	85	88,500	4.94694
L56 - S29	60	75	768,500	5.88564
L56 - S30	60	75	300,600	5.47799
L18 - S31	60	85	68,600	4.83632
L24 - S33	60	75	1,500,000	6.17609
L24 - S34	60	80	222,000	5.34635
L 7 - S35	60	80	95,500	4.98000
L 7 - S36	60	80	155,000	5.19033
L64 - S37	60	80	235,800	5.37254
L64 - S38	60	80	271,800	5.43425
L57 - S39	60	80	191,300	5.28171
L57 - S40	60	80	176,000	5.24551
L22 - S41	60	80	162,400	5.21059
L22 - S42	60	80	208,400	5.31890
L32 - S43	60	80	214,500	5.33143
L32 - S44	60	80	220,600	5.34361
L47 - S45	60	80	147,600	5.16909
L16 - S46	60	80	40,900	4.61172
L16 - S47	60	80	164,500	5.21617
L13 - S50	60	72.5	3,630,200[3]	......
L30 - S60	40	70	90,400	4.95617
L30 - S61	40	60	287,400	5.45849
L21 - S62	40	65	175,500	5.24428
L21 - S63	40	55	3,282,500[3]	......
L48 - S65	40	57.5	1,246,000	6.09552
L20 - S66	40	57.5	1,159,600	6.06432
L33 - S69	40	70	92,000	4.96379
L55 - S71	40	65	152,600	5.18355
L59 - S72	40	57.5	1,082,000	6.03423
L59 - S73	40	60	308,400	5.48911
L 6 - S74	40	57.5	561,000	5.74904
L 6 - S75	40	60	344,100	5.53668
L23 - S76	40	60	274,000	5.43775
L23 - S77	40	70	105,200	5.02202
L69 - S78	40	65	168,000	5.22531
L69 - S79	40	70	100,400	5.00173
L62 - S80	40	65	116,000	5.06446
L11 - S87	40	80	37,800	4.57749
L 8 - S105	40	75	36,500	4.56229
L67 - S108	40	75	54,000	4.73239
L27 - S118	40	55	5,375,000[3]	......
L26 - S122	60	72	652,800	5.81478
L53 - S124	60	72	1,873,500	6.27266
L29 - S130	40	57.5	591,000	5.77159
L29 - S131	40	60	573,000	5.75815
L71 - S132	40	65	126,000	5.10037
L71 - S133	40	70	71,000	4.85126
L31 - S134	40	60	359,000	5.55509
L31 - S135	40	70	76,000	4.88081
L37 - S136	40	65	174,000	5.24055
L37 - S137	40	57.5	715,000	5.85431

[1]Premature failure in grip. Not included in analysis.
[2]Failure at weldment. Not included in analysis.
[3]No failure.

grips, a careful inspection was made to determine whether the strand had rubbed against the steel front end block of the grip. In one test, *L1-S2*, this had occurred because of incorrect grip alignment during manufacture and caused a considerable decrease in fatigue life. This test is marked with a superscript 1 in Table 3 and is not included in the analysis of the results.

The fatigue life of specimen *L4-S6* was much lower than for other similar tests. An inspection of the failure section showed that fatigue had taken place in one of the wires in a region where a weldment had been made during manufacture of the strand. This test result, indicated by superscript 2 in Table 3, is also discarded in the analysis of the results.

Cumulative Damage Tests

Fatigue failure under cumulative damage loading was similar to constant cycle fatigue failure. However, actual wire fracture only took place during the application of overloadings. Even when several wires had already failed, further failures did not occur while loadings were being applied which were smaller than the fatigue limit.

The results of the cumulative damage tests are contained in Tables 5 and 6. A small number of the cumulative damage test specimens failed prematurely as a result of rubbing of the strand against the end block of the grip; the test results are given in tables, but are marked by asterisks and are not used in the analysis of the results.

ANALYSIS—CONSTANT CYCLE TESTS

Scatter is inherent in the results of all experimental work. It is present in the quantities being measured because of the variability of material properties; it is further introduced by imperfect methods of measurement and testing. Often the order of the scatter is small in comparison with the magnitude of the quantity being measured, in which case the quantity is adequately represented by the mean value. Thus, the static ultimate strength of the strand can be taken as 28.56 kips, the mean value of the test results in Table 2.

For other groups of tests, however, the deviations of results of similar tests from the mean value can be of the same order as the mean value itself. Such a situation has occurred in the constant cycle fatigue test

Table 4—Summary of Constant Cycle Strand Fatigue Test Data

Group	Stress Levels, % Static Ult.		No. of Replications	Fatigue Life			Log Fatigue Life		
	S_{min}	S_{max}		$\overline{N}$	D_N	$\frac{1}{\overline{N}} \cdot D_N$	$\overline{\log N}$	$\log^{-1}(\overline{\log N})$	D
A	40	70	6	89,200	13,400	0.1503	4.9460	88,300	0.0671
B	40	65	6	150,400	25,600	0.1705	5.1764	150,100	0.0768
C	40	60	6	357,700	121,900	0.3410	5.5392	346,100	0.1162
D	40	57.5	6	892,400	304,600	0.3410	5.9282	847,500	0.1548
E	60	85	6	81,900	13,620	0.1663	4.9084	80,980	0.0708
F	60	80	20	178,100	53,400	0.2998	5.2233	167,200	0.1793
G	60	75	7	705,630	421,900	0.5979	5.7827	606,300	0.2602

$\overline{N}$ = Mean fatigue life

D_N = Standard deviation of N

$\overline{\log N}$ = Mean of log N

D = Standard deviation of log N

data. For example in the Group *F* data in Table 4, the fatigue life observed in twenty replications of the same test varied between 235,000 cycles for specimen *L64-S37*, and 40,000 cycles for specimen *L16-S46*. Although a portion of the scatter in fatigue test results can always be attributed to experimental technique, it is now generally recognized that considerable variability is inherent in the phenomenon of fatigue failure[1].

With scatter of such magnitude in the results of similar tests, simple *S-N* curves and fatigue envelopes are clearly inadequate representations of fatigue properties. It is

Table 5
Strand Cumulative Damage Tests with Two Maximum Stress Levels

Test No.	Specimen No.	Stress Level, % Ult.			Block Shape		N_e	$\sum n/\bar{N}$	N_e/N_L
		S_{min}	S_{pred}	S_{01}	α	β/α			
3AA-1	L43-S48	60	65	85	30,000	0.25	357,300	1.08	1.10
3AA-2	L48-S64	60	65	85	30,000	0.25	385,700	1.15	1.20
3AB-1	L58-S85	60	65	85	300,000	0.25	221,000*	--	--
3AB-2	L45-S88	60	65	85	300,000	0.25	96,500*	--	--
3AB-3	L12-S92	60	65	85	300,000	0.25	540,000	1.11	1.67
3AB-4	L66-S91	60	65	85	300,000	0.25	550,000	1.24	1.70
3AC-1	L63-S100	60	65	85	10,000	0.25	349,000	1.07	1.08
3AC-2	L61-S103	60	65	85	10,000	0.25	390,000	1.20	1.20
3BA-1	L41-S55	60	70	85	30,000	0.25	148,000*	--	--
3BA-2	L68-S58	60	70	85	30,000	0.25	324,300	0.95	1.00
3BA-3	L40-S121	60	70	85	30,000	0.25	417,000	1.26	1.29
3CA-1	L54-S56	60	75	85	22,500	0.25	174,400	0.71	0.76
3CA-2	L68-S59	60	75	85	22,500	0.25	266,200	1.12	1.16
3DA-1	L43-S49	60	65	85	22,500	0.40	335,500	1.65	1.67
3DA-2	L55-S70	60	65	85	22,500	0.40	263,500	1.26	1.31
3EA-1	L54-S57	60	75	85	15,000	0.40	190,200	1.10	1.13
3EA-2	L20-S67	60	75	85	15,000	0.40	178,200	0.95	1.06
3FA-1	L44-S54	60	80	85	15,000	0.25	155,700	1.17	1.18
3FA-2	L33-S68	60	80	85	15,000	0.25	101,300	0.75	0.76
3FA-3	L52-S113	60	80	85	15,000	0.25	110,350	0.83	0.86
3FA-4	L52-S112	60	80	85	15,000	0.25	139,450	1.05	1.05
3FA-5	L34-S111	60	80	85	15,000	0.25	134,950	1.02	1.02
3FA-6	L34-S110	60	80	85	15,000	0.25	101,250	0.75	0.76
3FA-7	L35-S117	60	80	85	15,000	0.25	158,500	1.19	1.20
3FA-8	L42-S114	60	80	85	15,000	0.25	101,350	0.76	0.76
3FA-9	L35-S116	60	80	85	15,000	0.25	157,250	1.18	1.19
3FA-10	L42-S115	60	80	85	15,000	0.25	131,250	0.98	0.99
4AA-1	L11-S86	40	40	70	30,000	0.4	232,500	0.92	0.96
4AA-2	L66-S90	40	40	70	30,000	0.4	205,000	0.82	0.85
4BA-1	L14-S82	40	60	70	30,000	0.4	119,000	0.72	0.72
4BA-2	L45-S89	40	60	70	30,000	0.4	143,300	0.84	0.87
4BB-1	L62-S81	40	60	70	150,000	0.4	135,800	0.77	0.82
4BB-2	L58-S84	40	60	70	150,000	0.4	245,000	1.25	1.44
4BC-1	L14-S83	40	60	70	10,000	0.4	148,000	0.89	0.90
4BC-2	L12-S93	40	60	70	10,000	0.4	135,200	0.81	0.82

N_e = Observed fatigue life
N_L = Fatigue life predicted by Eq. 7
* = Failure in grip - not included in analysis

Table 6
Strand Cumulative Damage Tests with Three Maximum Stress Levels

Test No.	Specimen No.	Stress Level, % Ult.				Block Shape			N_e	$\sum n/\bar{N}$	N_e/N_L
		S_{min}	S_{pred}	S_{01}	S_{02}	α	β/α	γ/β			
5AA-1	L46-S98	60	60	80	85	60,000	0.4	0.4	295,700	0.97	1.01
5AA-2	L39-S107	60	60	80	85	60,000	0.4	0.4	238,600	0.80	0.81
5BA-1	L61-S102	60	75	80	85	30,000	0.4	0.4	235,200	1.00	1.04
5BA-2	L39-S106	60	75	80	85	30,000	0.4	0.4	250,000	1.07	1.10
5BA-3	L53-S125	60	75	80	85	30,000	0.4	0.4	48,100*	--	--
5CA-1	L63-S101	60	75	80	85	30,000	0.25	0.4	172,300	0.54	0.58
5CA-2	L 8-S104	60	75	80	85	30,000	0.25	0.4	232,500	0.74	0.78
5CA-3	L40-S120	60	75	80	85	30,000	0.25	0.4	147,500	0.48	0.50
5CA-4	L26-S123	60	75	80	85	30,000	0.25	0.4	82,500*	--	--
6AA-1	L60-S95	40	60	65	70	20,000	0.4	0.4	238,000	1.23	1.24
6AA-2	L19-S96	40	60	65	70	20,000	0.4	0.4	192,000	0.97	0.99
6BA-1	L60-S94	40	60	65	70	30,000	0.4	0.25	149,000	0.73	0.73
6BA-2	L19-S97	40	60	65	70	30,000	0.4	0.25	150,000	0.73	0.73
6BA-3	L27-S119	40	60	65	70	30,000	0.4	0.25	144,800	0.69	0.71
6CA-1	L46-S99	40	50	60	70	30,000	0.4	0.4	328,000	0.78	0.80
6CA-2	L67-S109	40	50	60	70	30,000	0.4	0.4	419,000	1.01	1.02

N_e = Observed fatigue life
N_L = Fatigue life predicted by Eq. 7
* = Failure in grip - not included in analysis

therefore necessary to associate variability with fatigue failure by treating the values of fatigue life observed in test replications as a sample taken from an infinite population of values which is distributed in some manner about a central or mean value and which is represented by some distribution function. Thus, we consider a probability of failure, P, varying between zero and unity, and with each value of P we associate a number, N, such that the probability is P that failure will occur at a number of cycles equal to or less than N.

Several investigations have been conducted to obtain information on the shape of frequency distributions associated with the phenomenon of fatigue failure. Müller-Stock[2] made 200 replications of a constant cycle fatigue test on steel specimens and obtained a distribution of lives having a pronounced skew with a long right hand tail. Freudenthal[1] obtained similar results and has shown, by a theoretical argument using several reasonable but approximate physical assumptions, that the distribution should be approximately logarithmic-normal.

Weibull[3] has suggested that although the log-normal distribution may fit test data well in the central region around the mean value, it may not represent extreme values very satisfactorily. In most cases test data are not extensive enough to provide information on the distribution at a distance from the mean value, and the log-normal distribution has been used in a number of recent investigations[4].

The log-normal distribution has the probability density function

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(X-\mu)^2}{2\sigma^2}} \qquad (1)$$

and cumulative distribution function

$$P = F(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{X} e^{\frac{-(X-\mu)^2}{2\sigma^2}} dX \quad (2)$$

where $X = \log N$, and μ and σ are the mean and standard deviation of the population of log N values. The functions $f(X)$ and $F(X)$ are completely determined when values for μ and σ have been obtained.

In order to investigate the suitability of the log-normal distribution to the constant cycle fatigue test results of this investigation a χ^2 goodness-of-fit test was conducted on the 20 replications of the Group F data.[5] The details of the χ^2 test are contained in Table 7. A χ^2 value of 1.2 was obtained which was well within the .05 significance level. A second χ^2 test was conducted using all of the test data contained in Table 3. The data for different load cycles were grouped together by

Table 7
χ^2 Goodness of Fit Test*
Constant Cycle Strand Fatigue Tests—Group F

Interval	0	E	0-E	$(0-E)^2$
$-\infty < Z < -0.675$	3	5	-2	4
$-0.675 \leq Z < 0$	6	5	1	1
$0 \leq Z < +0.675$	6	5	1	1
$+0.675 \leq Z < \infty$	5	5	0	0
$\sum$	20	20	0	6

$$Z = \frac{\log N - \overline{\log N}}{D}$$

0 = Observed number of test points within interval of Z values

E = Expected number of test points within interval of Z values

$$\chi^2 = \frac{\sum (0-E)^2}{E} = \frac{6}{5} = 1.20$$

For three (3) degrees of freedom, $\chi^2_{0.05} = 7.82$.

*The χ^2 test is described on page 85, Ref. 5.

making a change of variable from log N to Z, where

$$Z = \frac{\log N - \overline{\log N}}{D} \quad (3)$$

and log N and D are the mean and standard deviation of the set of data being grouped. This change of variable reduces each set of data to one with a mean of zero and standard deviation of unity. A plot of the grouped constant cycle fatigue test data is compared with the log-normal distribution in Fig. 5. The details of the χ^2 test for the grouped data are contained in Table 8. The χ^2 value of 10.70 is again well within the 0.05 significance level value. The assumption of a log-normal distribution will be made throughout this investigation.

In Fig. 6 fatigue life, N, has been plotted on logarithmic scale against maximum stress level for the constant cycle fatigue data. Although the tests were not designed primarily to indicate values of the fatigue limit, S_L, approximate values of 71

Table 8
χ^2 Goodness of Fit Test*
Constant Cycle Strand Fatigue Tests—Groups A through G

Interval	0	E	0-E	$(0-E)^2$
$-\infty < Z < -1.220$	4	6.33	-2.33	5.46
$-1.220 \leq Z < -0.766$	7	6.33	-0.67	.44
$-0.766 \leq Z < -0.430$	5	6.33	-1.33	1.78
$-0.430 \leq Z < -0.140$	4	6.33	-2.33	5.44
$-0.140 \leq Z < +0.140$	9	6.33	+2.67	7.12
$+0.140 \leq Z < +0.430$	4	6.33	-2.33	5.44
$+0.430 \leq Z < +0.766$	11	6.33	+4.67	21.70
$+0.766 \leq Z < +1.220$	10	6.33	+3.67	10.34
$1.220 \leq Z < +\infty$	3	6.33	-3.33	10.11
Σ	57	57	+5.33	67.83

$$Z = \frac{\log N - \overline{\log N}}{D}$$

0 = Observed number of test points within interval of Z values

E = Expected number of test points within interval of Z values

$$\chi^2 = \frac{\Sigma(0-E)^2}{E} = \frac{67.83}{E} = 10.07$$

For eight (8) degrees of freedom, $\chi^2_{0.05} = 15.51$

*The χ^2 test is described on page 85, Ref. 5.

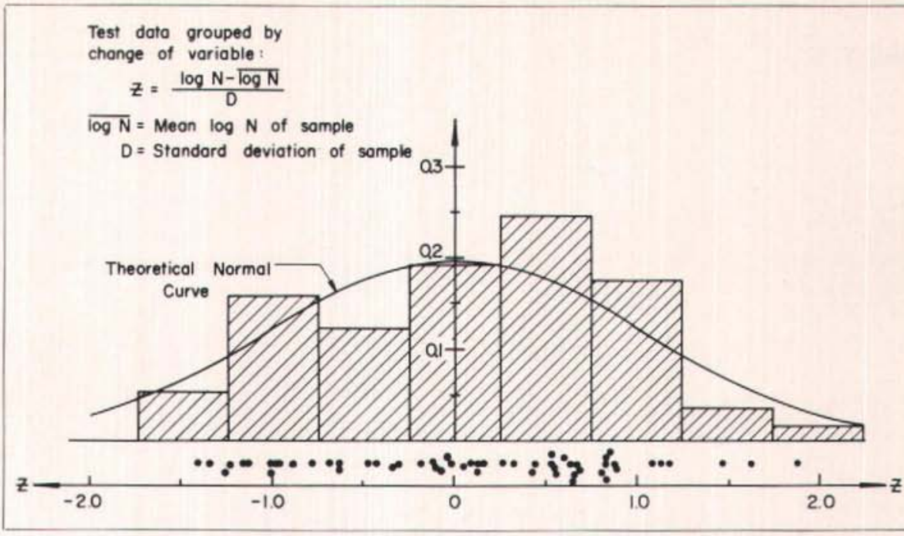

Fig. 5—Frequency Distribution of Grouped Constant Cycle Test Data

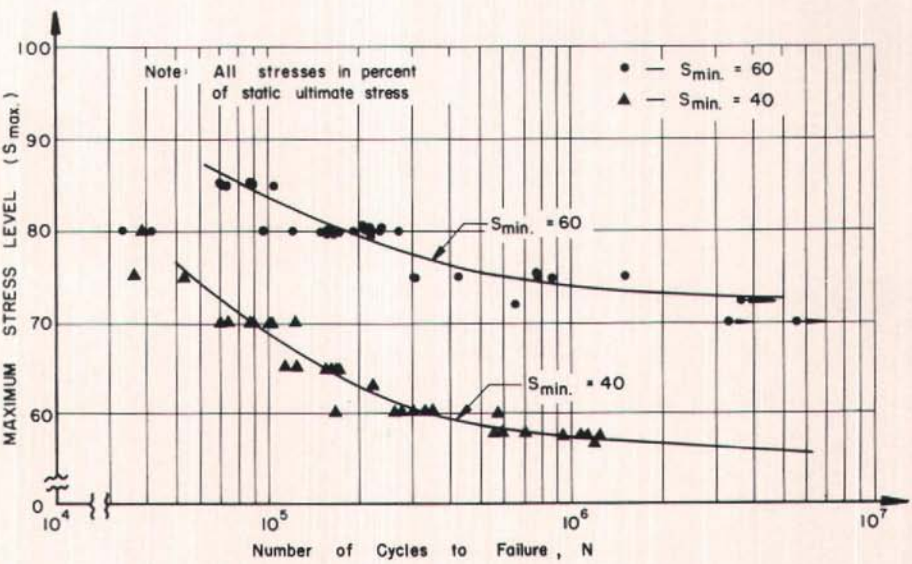

Fig. 6—Maximum Stress Level Versus Fatigue Life, Constant Cycle Tests

and 55 percent have been obtained for the 60 and 40 percent minimum stress levels, respectively, by extrapolation.

In Fig. 7 the two sets of data have been plotted together using variables $R = (S_{max} - S_L)$ and log N. A mean line has been fitted to this data by using a relation of the form

$$\overline{\log N} = \frac{C_1}{R} + C_2 + C_3 R$$

The method of least squares was used to obtain the following three simultaneous equations for the evaluation of the open parameters C_1,

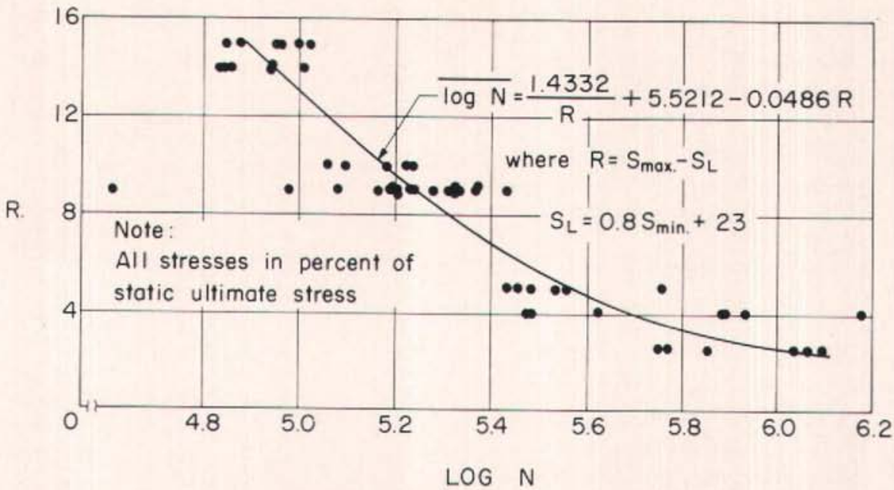

Fig. 7—R Versus Log N

C_2, and C_3;

$$C_1 \sum_{i=1}^{n} \frac{1}{R_i^2} + C_2 \sum_{i=1}^{n} \frac{1}{R_i} + nC_3 = \sum_{i=1}^{n} \frac{\log N_i}{R_i}$$

$$C_1 \sum_{i=1}^{n} \frac{1}{R_i} + nC_2 + C_3 \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \log N_i$$

$$nC_1 + C_2 \sum_{i=1}^{n} R_i + C_3 \sum_{i=1}^{n} R_i^2 = \sum_{i=1}^{n} \log N_i R_i$$

Solution of these equations yields the relation

$$\overline{\log N} = \frac{1.4332}{R} + 5.5212 - 0.0486\,R \quad (4)$$

where $R = S_{max} - S_L$.

Assuming a linear variation of S_L between the 40 and 60 percent minimum stress level values of 55 and 71 percent, the following equation is obtained for the fatigue limit;

$$S_L = 0.8\,S_{min} + 23 \quad (5)$$

Eq. 4 and 5 provide values for the mean fatigue life corresponding to any stress amplitude in the region under consideration. It is of course possible, and in some situations may be more convenient, to obtain the mean *S-N* curve corresponding to any minimum stress level between 40 and 60 percent directly from Fig. 6 by linear interpolation.

Since values of both the mean and standard deviation are required to specify completely the log-normal frequency distribution, it is now necessary to obtain appropriate values for standard deviation corresponding to each stress amplitude.

The best unbiased estimate of the standard deviation of the population is given by

$$D = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (\log N - \overline{\log N})^2}$$

where n is the number of replications and $\overline{\log N}$ is the mean value for the sample. Values of D for the seven sets of test data are plotted against R in Fig. 8. Considerable variation occurs among the points. A change in the position of the test specimens in the loading rig from center to quarter-point reduced the scatter of the 40 percent minimum stress level test results quite considerably; the larger values corresponding to the center-point load position are probably due in part to larger experimental errors associated with that set-up. However, a fairly consistent trend is followed and both the quarter-point and center-point set-up data yield reasonably linear variations of D with R.

The use of anything but the simplest relation is unwarranted by the test data available, and for the purposes of this investigation, a straight line variation is assumed and fitted to the seven points. A least squares fit yields for the standard deviation,

$$D = 0.2196 - 0.0103R \qquad (6)$$

The S-N-P (maximum stress–number of cycles–probability of failure) relation is thus given by the equations

$$P = F(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{X} e^{\frac{-(X-\mu)^2}{2\sigma^2}}\, dX \qquad (2)$$

$$X = \log N;$$

$$\mu = \overline{\log N} = \frac{1.4332}{R} + 5.5212 - 0.0486R \qquad (4)$$

$$R = S_{max} - (0.8\, S_{min} + 23)$$

$$\sigma = D = 0.2196 - 0.0103R \qquad (6)$$

Values of P corresponding to values of X in Eq. 2, and vice versa, can of course be obtained most easily from standard tables[5].

It should be noted that the above equations have been derived for the following ranges of variables;

$$40 \leqq S_{min} \leqq 60$$
$$0 < R \leqq 15$$

ANALYSIS—CUMULATIVE DAMAGE TESTS

A general, quantitative theory of fatigue failure must obviously be based on assumptions which describe, at least approximately, the fundamental physical and metallur-

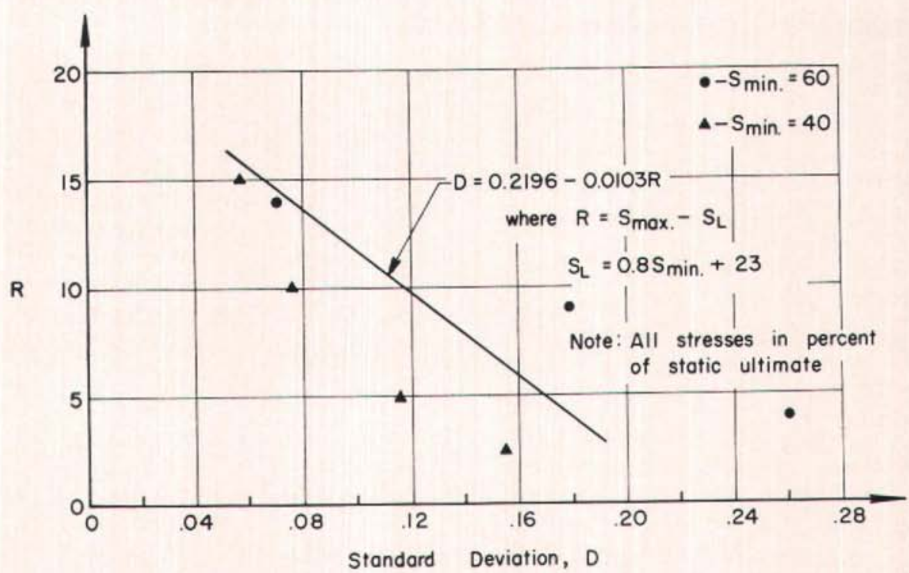

Fig. 8—R Versus Standard Deviation

gical changes which take place in a material subjected to fatigue loading. However, investigators are not yet in agreement on the essential nature of the fatigue failure mechanism, nor on general principles which yield quantitative data on fatigue life. Extensive metallurgical studies will be necessary before satisfactory progress can be made towards this goal. Quantitative information on the fatigue properties of materials must therefore come at present from engineering studies which are phenomenological and experimental in nature, and hence restricted in application.

Mean Fatigue Life under Varied Repeated Loadings

One of the simplest and most widely known procedures for predicting mean fatigue life under varied repeated loadings was suggested by Palmgren[6] and later by Miner[7]. In this approach the cycle ratio, r_i, is defined for a stress amplitude S_i as

$$r_i = n_i/\overline{N}_i$$

where n_i is the number of cycles of S_i loading which have been applied to the specimen, and N_i is the mean fatigue life corresponding to S_i. It is assumed that fatigue damage accumulates in the specimen in direct proportion to the sum of the cycle ratios. Damage is complete and failure takes place when the summation is equal to unity, i.e., for q different stress amplitudes, when

$$\sum_{i=1}^{q} r_i = \sum_{1}^{q} n_i/\overline{N}_i = 1 \qquad (7)$$

Two series of tests conducted by Miner on aluminum alloy specimens yielded mean summation values of 1.05 and 0.98, with extreme values of 1.49 and 0.61. However, tests conducted by other investigators[8] have in some cases yielded results differing considerably from unity.

In Tables 5 and 6 a comparison is made between the fatigue lives observed in the tests and values predicted by Eq. 7. The values of $\Sigma n/\overline{N}$ observed in the tests were quite close to unity, indicating reasonable agreement between experimental and predicted values. The mean of the $\Sigma n/\overline{N}$ values for all tests is 0.97, with extreme values of 0.48 and 1.65 and a standrad deviation of 0.224. This value for the standard deviation is quite comparable to the values given in Table 4 for the standard deviation of the quantity $N/\overline{N}$ obtained in the constant cycle tests. Since

$$n/\overline{N}_{experimental} \approx N_e/\overline{N}_L$$

where N_e is the observed fatigue life and N_L is the value predicted by the linear summation theory, and variabilities of the cumulative damage tests and constant cycle tests, as measured by the standard deviations, are of a similar order. It therefore appears reasonable to attribute the observed scatter in $\Sigma n/\overline{N}$ to inherent variability in the test data rather than to inapplicability of the theory.

It has been suggested in some recent cumulative damage studies[9,10] that there may exist an "interaction" effect between the fatigue damage caused by repeated load cycles of different magnitude. Accordingly, intermittent high-stress cycles can accelerate the fatigue damage caused by low-stress cycles and hence have a far more severe effect on fatigue life than would be indicated by Miner's linear equation, which is based on the assumption that the rate of fatigue damage at one stress level is independent of the application of other stress levels.

It will be noted that there is no evidence at all in the test data of the damaging effect of understresses when mixed with intermittent overloadings. On the contrary, there is a slight tendency for understresses i.e. stresses lower than the fatigue limit, to improve fatigue resistance. This may be seen in the results of tests *3AA*, *3AB*, *3AC*, *3BA*, and *3DA* which are contained in Table 5, where the summation values are always a little above unity. That this improvement is actually due to the presence of the understresses and is not simply the beneficial effect of intermittent application of the overloads is indicated by tests *4AA* and *5AA*. These tests, in which the understresses are of zero amplitude—i.e. correspond to rest periods—gave summation values slightly less than unity. The evidence is of course insufficient to establish a definite trend of improved fatigue life with the presence of understresses, however, it does seem reasonable to assume in the following that understresses will not contribute to fatigue damage.

Although no interaction effect can be observed between high and low stress levels, tests *5CA* and *6BA*, in which the stress blocks contain three different overstresses, yield summation values considerably less than unity and might indicate an interaction effect. However, the two other tests with three overstresses, *5BA* and *6AA*, both have summation values greater than unity. No definite trend is therefore indicated.

In view of the very reasonable agreement between test results and values predicted by the linear theory, Eq. 7 will be used in this investigation for the prediction of mean fatigue life of strand reinforcement under varying cycles of repeated loading.

Probable Fatigue Life under Varied Cycles of Repeated Loading

It was seen earlier that fatigue life under constant cycle loading is distributed log-normally, at least to first approximation, about the mean value. It seems reasonable to expect a log-normal distribution to apply approximately also to fatigue life under varied load cycles. If the log-normal assumption were made, probable fatigue life would be established by the value of the mean fatigue life, given by Eq. 7, together with a value which would have to be estimated for the standard deviation. However, instead of assuming a log-normal distribution and proceeding to study possible methods of estimating the standard deviation, a direct approach is made in the following by generalizing the linear accumulation theory so that it may be applied at all probability levels.

Considering a load history which consists of two stress levels, S_1 and S_2, occurring in the proportions a and $(1\text{-}a)$, the mean fatigue life of a strand is given by the equation

$$\Sigma\, n/N = 1$$

or

$$\frac{a\,N\,(0.5)}{N_1\,(0.5)} + \frac{(1\text{-}a)\,N\,(0.5)}{N_2\,(0.5)} = 1 \qquad (8)$$

where $N_1\,(0.5)$, $N_2\,(0.5)$, and $N\,(0.5)$ are the mean fatigue lives corresponding to S_1, S_2 and the combined loading respectively.

In general, considering possible conditions where the linear accumulation theory may not be satisfactory, the cumulative damage theory for mean fatigue life would provide a relation of the form

$$\Phi\,[N\,(0.5), a] = 0 \qquad (9)$$

where $0 \leqq a \leqq 1$. Eq. 9 describes a relation between $N(0.5)$ and a, as shown in Fig. 9. However, the fatigue lives corresponding to S_1 and S_2 actually consist of distribution functions with ranges of N_1 and N_2 values corresponding to different probability levels, as shown also in Fig. 9. In order to obtain curves corresponding to probability levels other than 0.5, it appears reasonable to assume that the form of the N-a relation will not alter with the probability level, and that Eq. 9 may be generalized to

$$\Phi\,[N(P), a] = 0 \qquad (10)$$

to apply to all probability levels. It will be noted that although the fatigue lives at S_1 and S_2 may be log-normally distributed, the distribution obtained from Eq. 10 for values of a other than zero and unity will not, in general, be log-normal.

Eq. 8 may be rearranged as

$$N(0.5)\left\{a\,[N_2(0.5) - N_1(0.5)] + N_1(0.5)\right\} - N_1(0.5)\cdot N_2(0.5) = 0 \qquad (11)$$

and generalized according to Eq. 10 to

$$N(P)\left\{a\,[N_2(p) - N_1(P)] + N_1(P)\right\} - N_1(P)\cdot N_2(P) = 0 \qquad (12)$$

Eq. 12 allows the fatigue life to be determined for any probability level and any combination of S_1 and S_2. In Fig. 10 a diagram has been constructed similar to Fig. 9 using Eq. 12 and N values corresponding to a 60 percent minimum stress level and 80 and 85 percent values for S_1 and S_2. These load cycles were used in cumulative damage test $3FA$ as S_{pred} and S_{01} respectively, the results of which are contained in Table 5. The predicted cumulative frequency distribution is compared with the distribution of the ten test replications in Fig. 10.

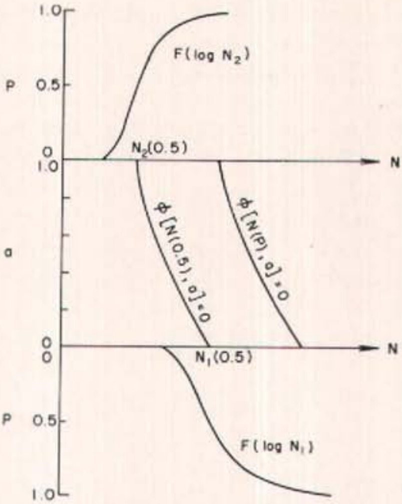

Fig. 9—Cumulative Damage Theory at Probability Level P

This number of test replications is of course too small to provide justification for the generalization from Eq. 9 to Eq. 10, but in view of the complete lack of other test data, the reasonableness and simplicity of the procedure, and the correlation between these few tests and the predicted distribution, it will be adopted here.

When q different stress levels are combined with relative frequencies of occurrence a_i, Eq. 7 may be generalized in the above manner to yield

$$\sum_{i=1}^{q} \frac{a_i\,N(P)}{N_i(P)} = 1$$

or

$$N(P) = \frac{1}{\sum_{i=1}^{q} \frac{a_i}{N_i(P)}} \qquad (13)$$

for any probability level P. Eq. 13

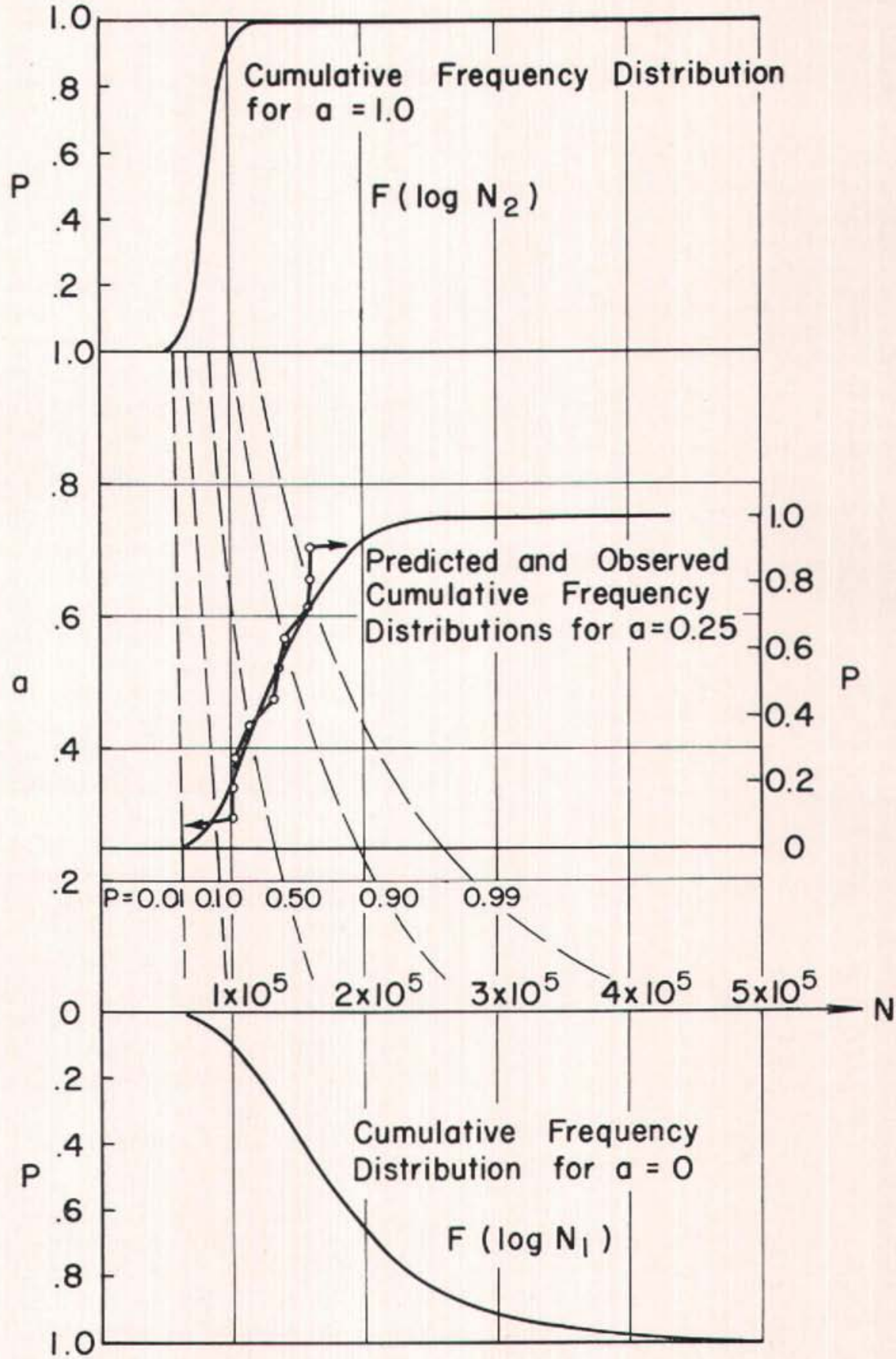

Fig. 10—Predicted and Observed Frequency Distributions for Cumulative Damage Test

will be used in this investigation, together with the constant cycle *S-N-P* relation, represented by Eqs. 2, 4, and 6, to predict the probable fatigue life of strand specimens.

CONCLUDING REMARKS

It has already been noted that quantitative information on material fatigue properties must at present come from experimental studies, and therefore that such information is restricted in application. It is important to emphasize the limited applicability of the strand fatigue test data obtained in this investigation. All of the strand tests were conducted on unrusted 7/16 in. diameter strand from one manufacturer. Strand which has been stored for some time and allowed to rust will have poorer fatigue properties; some differences in the fatigue properties of strand of different sizes must also be expected. Considerable variation might also be expected between the products of different manufacturers, and, quite possibly, in the product of one manufacturer over a period of time.

More fatigue tests are obviously required to investigate each of these effects. Such fatigue tests may well indicate the advisability of using an equation for mean fatigue life more conservative than Eq. 4; they most certainly will indicate values for standard deviation greater than those represented by Eq. 6.

The experimental study described above was concerned with the fatigue properties of the strand in the life region between 50,000 cycles and 5 million cycles. Approximate values for fatigue limit were adopted, on the basis of an extrapolation of the mean *S-N* curves, and were used in the derivation of Eq. 4. Some error in the values of the mean fatigue limit does not however influence significantly the *fit* of Eq. 4 in the finite life region under consideration. A different type of test[4] would of course be required to establish accurate values for the probable fatigue limit of the material.

The results of the cumulative damage tests showed good correlation with mean fatigue life predicted by the linear theory proposed by Palmgren and Miner. A generalized form of the linear theory has been developed to apply to all probability levels. The cumulative damage tests indicated that stress cycles in the loading history which are smaller than the fatigue limit will not contribute to fatigue failure in the strand. Thus, beam loadings which cause flexural cracks to open should not shorten beam fatigue life provided the stresses induced in the strand reinforcement are smaller than the fatigue limit.

It is emphasized that the cumulative damage tests were conducted in such a manner that the different load levels were distributed more or less evenly throughout the life of the specimen. The above conclusions therefore may not apply to the more usual laboratory fatigue test in which all of the loadings of a given magnitude are applied to the specimen consecutively. The testing method used in this investigation should correspond more closely to conditions in an actual structure.

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REFERENCES

1. Freudenthal, A. M., "Planning and Interpretation of Fatigue Tests," Symposium and Statistical Aspects of Fatigue, Special Technical Publication No. 121, ASTM, 1952.
2. Müller-Stock, H., Der Einfluss Dauernd und Unterbrochen Wirkender, Schwingender Uberbeanspruchung auf die Entwicklung des Dauerbruchs, Mitt Kohle und Eisenforsch, Bd. 2, 1938, pp. 83-107.
3. Weibull, W., "Discussion of Reference 2", Symposium on Statistical Aspects of Fatigue, Special Technical Publication No. 121, ASTM, 1952.
4. Grover, H. J., Gordon, S. A., Jackson, L. R., *Fatigue of Metals and Structures*, U.S. Government Printing Office, Revised Edition, June 1960.
5. Crow, E. L., Davis, F. A., Maxfield, M. W., *Statistics Manual*, Dover Publications, New York, 1960.
6. Palmgren, A., "Die Lebensdauer Von Kugellargern" *Ver. Duet. Ingenieure*, Vol. 68, No. 14, 1924, pp. 339-341.
7. Miner, M. A., "Cumulative Damage in Fatigue", *Journal of Applied Mechanics*, Vol. 12, P.A.-159, September 1945.
8. Dolan, T. J., Richart, F. E., Work, C. E., "Influence of Fluctuations in Stress Amplitude on the Fatigue of Metals", *Proceedings*, ASTM, Vol. 49, 1949.
9. Freudenthal, A. M., "Fatigue of Structural Metals Under Random Loading", Special Publication No. 284, ASTM Symposium on Acoustical Fatigue, 1960.
10. Coffin, L. F., "Discussion of Reference 10", Special Publication No. 284, ASTM Symposium on Acoustical Fatigue, 1960.

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